Sample Quiz 8

Sample Quiz 8, Problem 1. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order $N$, for example equations like

\[ y'' + 16y = 0, \quad y''' + 4y'' = 0, \quad \frac{d^5 y}{dx^5} + 2y''' + y'' = 0. \]

1. Find the $N$th degree characteristic equation by Euler’s substitution $y = e^{rx}$. For instance, $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$, a polynomial equation of degree $N = 2$.

2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the $N$ roots according to multiplicity.

3. Construct $N$ distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.

The solution space $S$ of the differential equation is given by

\[ S = \text{span}(\text{the } N \text{ Euler solution atoms}). \]

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2 e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms $e^{0x}, xe^{0x}, e^{2x}$.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x} \cos(3x), e^{2x} \sin(3x)$.

Four complex conjugate roots listed according to multiplicity as $2 \pm 3i, 2 \pm 3i$ produce four atoms $e^{2x} \cos(3x), e^{2x} \sin(3x), xe^{2x} \cos(3x), xe^{2x} \sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2 e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a \pm bi$ ($b > 0$) arising from roots of $(r-a)^2 + b^2 = 0$ produce two atoms $e^{ax} \cos(bx), e^{ax} \sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) \[ y'' + 16y = 0 \]
(b) \[ y'' + 16y = 0 \]
(c) \[ y''' + 16y'' = 0 \]
(d) \[ y'' + 16y = 0, \quad y(0) = 1, \quad y'(0) = -1 \]
(e) \[ y''' + 9y'' = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1 \]
(f) The characteristic equation is $(r - 2)^2(r^2 - 4) = 0$.
(g) The characteristic equation is $(r - 1)^2(r^2 - 1)((r + 2)^2 + 4) = 0$.
(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3 + 4i, 3 - 4i$.

\[ ^1 \text{The Reason: } \cos(3x) = \frac{1}{2} e^{3xi} + \frac{1}{2} e^{-3xi} \text{ by Euler’s formula } e^{i\theta} = \cos \theta + i \sin \theta. \text{ Then } e^{2x} \cos(3x) = \frac{1}{2} e^{2x+3xi} + \frac{1}{2} e^{2x-3xi} \text{ is a linear combination of exponentials } e^{rx} \text{ where } r \text{ is a root of the characteristic equation. Euler’s substitution implies } e^{rx} \text{ is a solution, so by superposition, so also is } e^{2x} \cos(3x). \text{ Similar for } e^{2x} \sin(3x). \]
Solutions to Problem 1

(a) $y'' + 16y' = 0$ upon substitution of $y = e^{rx}$ becomes $(r^2 + 16)e^{rx} = 0$. Cancel $e^{rx}$ to find the characteristic equation $r^2 + 16r = 0$. It factors into $r(r + 16) = 0$, then the two roots $r$ make the list $r = 0, -16$. The Euler solution atoms for these roots are $e^{0x}, e^{-16x}$. Report the general solution $y = c_1 e^{0x} + c_2 e^{-16x} = c_1 + c_2 e^{-16x}$, where symbols $c_1, c_2$ stand for arbitrary constants.

(b) $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$. Because a quadratic equation $(r - a)^2 + b^2 = 0$ has roots $r = a \pm bi$, then the root list for $r^2 + 16 = 0$ is $0 + 4i, 0 - 4i$, or briefly $\pm 4i$. The Euler solution atoms are $e^{0x} \cos(4x), e^{0x} \sin(4x)$. The general solution is $y = c_1 \cos(4x) + c_2 \sin(4x)$, because $e^{0x} = 1$.

(c) $y'''' + 16y'' = 0$ has characteristic equation $r^4 + 4r^2 = 0$ which factors into $r^2(r^2 + 16) = 0$ having root list $0, 0, 0 \pm 4i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x} \cos(4x), e^{0x} \sin(4x)$. Then the general solution is $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 \cos(4x) + c_4 \sin(4x)$.

(d) $y'' + 16y = 0$, $y(0) = 1$, $y'(0) = -1$ defines a particular solution $y$. The usual arbitrary constants $c_1, c_2$ are determined by the initial conditions. From part (b), $y = c_1 \cos(4x) + c_2 \sin(4x)$. Then $y' = -4c_1 \sin(4x) + 4c_2 \cos(4x)$. Initial conditions $y(0) = 1, y'(0) = -1$ imply the equations $c_1 \cos(0) + c_2 \sin(0) = 1, -4c_1 \sin(0) + 4c_2 \cos(0) = -1$. Using $\cos(0) = 1$ and $\sin(0) = 0$ simplifies the equations to $c_1 = 1$ and $4c_2 = -1$. Then the particular solution is $y = c_1 \cos(4x) + c_2 \sin(4x) = \cos(4x) - \frac{1}{4} \sin(4x)$.

(e) $y'''' + 9y'' = 0$, $y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$ is solved like part (d). First, the characteristic equation $r^4 + 9r^2 = 0$ is factored into $r^2(r^2 + 9) = 0$ to find the root list $0, 0, 0 \pm 3i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x} \cos(3x), e^{0x} \sin(3x)$, which implies the general solution $y = c_1 + c_2 x + c_3 \cos(3x) + c_4 \sin(3x)$. We have to find the derivatives of $y$: $y' = c_2 - 3c_3 \sin(3x) + 3c_4 \cos(3x), y'' = -9c_3 \cos(3x) - 9c_4 \sin(3x), y''' = 27c_3 \sin(3x) - 27c_4 \cos(3x)$. The initial conditions give four equations in four unknowns $c_1, c_2, c_3, c_4$:

$$
c_1 + c_2(0) + c_3 \cos(0) + c_4 \sin(0) = 0,
2c_2 - 3c_3 \sin(0) + 3c_4 \cos(0) = 0,
-9c_3 \cos(0) - 9c_4 \sin(0) = 1,
27c_3 \sin(0) - 27c_4 \cos(0) = 1,
$$

which has invertible coefficient matrix.

(f) The characteristic equation is $(r - 2)^2(r^2 - 4) = 0$. Then $(r - 2)^3(r + 2) = 0$ with root list 2, 2, 2, −2 and Euler atoms $e^{2x}, xe^{2x}, x^2 e^{2x}, e^{-2x}$. The general solution is a linear combination of these four atoms.

(g) The characteristic equation is $(r - 1)^2(r^2 - 1)((r + 2)^2 + 4) = 0$. The root list is 1, 1, 1, −1, −2± 2i with Euler atoms $e^{x}, xe^{x}, x^2 e^{x}, e^{-x}, e^{-2x} \cos(2x), e^{-2x} \sin(2x)$. The general solution is a linear combination of these six atoms.

(h) The characteristic equation roots, listed according to multiplicity, are 0, 0, 0, −1, 2, 2, 3 + 4i, 3 − 4i. Then the Euler solution atoms are $e^{0x}, xe^{0x}, x^2 e^{0x}, e^{-x}, e^{2x}, xe^{2x}, x^3 \cos(4x), x^3 \sin(4x)$. The general solution is a linear combination of these eight atoms.
Sample Quiz 8, Problem 2. Laplace Theory

Laplace theory implements the method of quadrature for higher order differential equations, linear systems of differential equations, and certain partial differential equations.

Laplace’s method solves differential equations.

The Problem. Solve by table methods or Laplace’s method.

(a) Forward table. Find \( \mathcal{L}(f(t)) \) for \( f(t) = te^{2t} + 2t \sin(3t) + 3e^{-t} \cos(4t) \).

(b) Backward table. Find \( f(t) \) for

\[
\mathcal{L}(f(t)) = \frac{16}{s^2 + 4} + \frac{s + 1}{s^2 - 2s + 10} + \frac{2}{s^2 + 16}.
\]

(c) Solve the initial value problem \( x''(t) + 256x(t) = 1, \ x(0) = 1, \ x'(0) = 0 \).

Solution (a).

\[
\mathcal{L}(f(t)) = \mathcal{L}(te^{2t}) + 2\mathcal{L}(t \sin(3t)) + 3\mathcal{L}(e^{-t} \cos(4t))
\]

\[
= -\frac{d}{ds}\mathcal{L}(e^{2t}) - \frac{2}{ds}\mathcal{L}(\sin(3t)) + 3\mathcal{L}(e^{-t} \cos(4t)) \quad \text{Differentiation rule}
\]

\[
= -\frac{d}{ds}\mathcal{L}(e^{2t}) - \frac{2}{ds}\mathcal{L}(\sin(3t)) + 3\mathcal{L}(\cos(4t)) \bigg|_{s=s+1} \quad \text{Shift rule}
\]

\[
= -\frac{d}{ds}\left(\frac{1}{s-2}\right) - \frac{2}{ds}\left(\frac{3}{s^2+9}\right) + 3\frac{s}{s^2+16} \bigg|_{s=s+1} \quad \text{Calculus}
\]

Solution (b).

\[
\mathcal{L}(f(t)) = \frac{16}{(s-2)^2} + \frac{s+1}{(s-2)^2+10} + \frac{2}{(s-2)^2+16}
\]

\[
= 8\frac{s+1}{(s-2)^2} + \frac{2}{(s-2)^2+16} \quad \text{Prep for backward table}
\]

\[
= 8\mathcal{L}(\sin 2t) + \frac{3}{(s-2)^2+9} + \frac{1}{2}\mathcal{L}(\sin 4t) \quad \text{backward table}
\]

\[
= 8\mathcal{L}(\sin 2t) + \mathcal{L}(\cos 3t + \frac{2}{3} \sin 3t) \bigg|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) \quad \text{shift rule}
\]

\[
= 8\mathcal{L}(\sin 2t) + \mathcal{L}(e^t \cos 3t + e^{\frac{2}{3}} t \sin 3t) + \frac{1}{2}\mathcal{L}(\sin 4t) \quad \text{shift rule}
\]

\[
= 8\mathcal{L}(\sin 2t) + e^t \cos 3t + e^{\frac{2}{3}} t \sin 3t + \frac{3}{2} \sin 4t \quad \text{Linearity}
\]

\[
f(t) = 8\sin 2t + e^t \cos 3t + e^{\frac{2}{3}} t \sin 3t + \frac{3}{2} \sin 4t \quad \text{Lerch’s cancel rule}
\]

Solution (c).

\[
\mathcal{L}(x''(t) + 256x(t)) = \mathcal{L}(1) \quad \text{\mathcal{L} acts like matrix mult}
\]

\[
s\mathcal{L}(x') - x'(0) + 256\mathcal{L}(x) = \mathcal{L}(1) \quad \text{\mathcal{L} acts like matrix mult}
\]

\[
s(s\mathcal{L}(x) - x(0)) - x'(0) + 256\mathcal{L}(x) = \mathcal{L}(1) \quad \text{Parts rule}
\]

\[
s^2\mathcal{L}(x) - s + 256\mathcal{L}(x) = \mathcal{L}(1) \quad \text{Use } x(0) = 1, x'(0) = 0
\]

\[
(s^2 + 256)\mathcal{L}(x) = s + \mathcal{L}(1) \quad \text{Collect } \mathcal{L}(x) \text{ left}
\]

\[
\mathcal{L}(x) = \frac{s+1}{(s^2+256)} \quad \text{Isolate } \mathcal{L}(x) \text{ left}
\]

\[
\mathcal{L}(x) = \frac{1}{s+1} \quad \text{Forward table}
\]

\[
\mathcal{L}(x) = \frac{s^2+1}{s(s^2+256)} \quad \text{Algebra}
\]

\[
\mathcal{L}(x) = \frac{A}{s} + \frac{B}{s+256} \quad \text{Partial fractions}
\]

\[
\mathcal{L}(x) = A\mathcal{L}(1) + B\mathcal{L}(\cos 16t) + \frac{C}{16}\mathcal{L}(\sin 16t) \quad \text{Backward table}
\]

\[
\mathcal{L}(x) = \mathcal{L}(A + B \cos 16t + \frac{C}{16} \sin 16t) \quad \text{Linearity}
\]

\[
x(t) = A + B \cos 16t + \frac{C}{16} \sin 16t \quad \text{Lerch’s rule}
\]
The partial fraction problem remains:

\[
\frac{s^2 + 1}{s(s^2 + 256)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 256}
\]

This problem is solved by clearing the fractions, then swapping sides of the equation, to obtain

\[A(s^2 + 256) + (Bs + C)(s) = s^2 + 1.\]

Substitute three values for \(s\) to find 3 equations in 3 unknowns \(A, B, C\):

\[
\begin{align*}
  s &= 0 & 256A &= 1 \\
  s &= 1 & 257A + B + C &= 2 \\
  s &= -1 & 257A + B - C &= 2
\end{align*}
\]

Then \(A = 1/256, B = 255/256, C = 0\) and finally

\[x(t) = A + B \cos 16t + \frac{C}{16} \sin 16t = \frac{1 + 255 \cos 16t}{256}\]

**Answer Checks**

# Sample quiz 8
# answer check problem 2(a)
f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
with(inttrans): # load laplace package
laplace(f,t,s);
# The last two fractions simplify to 3(s+1)/((s+1)^2+16).
# answer check problem 2(b)
F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
invlaplace(F,s,t);
# answer check problem 2(c)
de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
dsolve([de,ic],x(t));
# answer check problem 2(c), partial fractions
convert((s^2+1)/(s*(s^2+256)),parfrac,s);

The output appears on the next page
# Sample quiz 11
# answer check problem 2(a)
\[ f := t \cdot e^{2t} + 2t \cdot \sin(3t) + 3t \cdot \exp(-t) \cdot \cos(4t); \]
\[ f^2 := t^2 e^{2t} + 2t \sin(3t) + 3e^{-t} \cos(4t) \]  
(1)

with(inttrans): \# load laplace package
laplace(f,t,s) assuming s::real;
\[ \frac{1}{(s - 2)^2} + \frac{12s}{(s^2 + 9)^2} + \frac{3}{2(s + 1 - 4i)} + \frac{3}{2(s + 1 + 4i)} \]  
(2)

# The last two fractions simplify to 3(s+1)/((s+1)^2+16).

# answer check problem 2(b)
\[ F := \frac{16}{s^2 + 4} + \frac{s + 1}{s^2 - 2s + 10} + \frac{2}{s^2 + 16}; \]
\[ F := \frac{16}{s^2 + 4} + \frac{s + 1}{s^2 - 2s + 10} + \frac{2}{s^2 + 16} \]  
(3)

invlaplace(F,s,t);
\[ 8 \sin(2t) + \frac{1}{2} \sin(4t) + \frac{1}{3} e^t (3 \cos(3t) + 2 \sin(3t)) \]  
(4)

# answer check problem 2(c)
de:=diff(x(t),t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
\[ de := \frac{d^2}{dt^2} x(t) + 256 x(t) = 1 \]
\[ ic := x(0) = 1, D(x)(0) = 0 \]  
(5)
dsolve([de,ic],x(t));
\[ x(t) = \frac{1}{256} + \frac{255}{256} \cos(16t) \]  
(6)

# answer check problem 2(c), partial fractions
convert((s^2+1)/(s*(s^2+256)),parfrac,s);
\[ \frac{1}{256} s + \frac{255}{256} \frac{s}{s^2 + 256} \]  
(7)