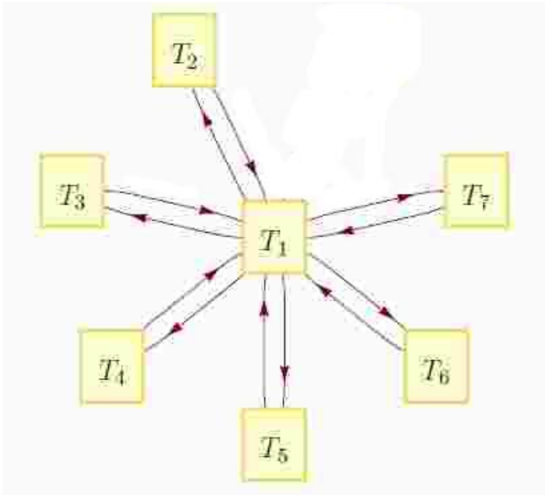
# Sample Quiz 13

## Problem 1.




Flow through each pipe is $f$ gallons per unit time.

Each tank has constant volume $V$.

Symbols $x_1(t)$ to $x_7(t)$ are the salt amounts in tanks $T_1$ to $T_7$, respectively.

The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate $f$ has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$\begin{aligned}
x_1'(t) &= \tfrac{f}{V}\left(x_2(t)+x_3(t)+x_4(t)+x_5(t)+x_6(t)+x_7(t)-6x_1(t)\right)\\
x_2'(t) &= \tfrac{f}{V}\left(x_1(t)-x_2(t)\right),\\
x_3'(t) &= \tfrac{f}{V}\left(x_1(t)-x_3(t)\right),\\
x_4'(t) &= \tfrac{f}{V}\left(x_1(t)-x_4(t)\right),\\
x_5'(t) &= \tfrac{f}{V}\left(x_1(t)-x_5(t)\right),\\
x_6'(t) &= \tfrac{f}{V}\left(x_1(t)-x_6(t)\right),\\
x_7'(t) &= \tfrac{f}{V}\left(x_1(t)-x_7(t)\right).
\end{aligned}$$

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**Problem 1(a).** Change variables $t = Vr/f$ to obtain the new system

$$\begin{aligned}
\frac{dx_1}{dr} &= x_2+x_3+x_4+x_5+x_6+x_7-6x_1\\
\frac{dx_2}{dr} &= x_1-x_2,\\
\frac{dx_3}{dr} &= x_1-x_3,\\
\frac{dx_4}{dr} &= x_1-x_4,\\
\frac{dx_5}{dr} &= x_1-x_5,\\
\frac{dx_6}{dr} &= x_1-x_6,\\
\frac{dx_7}{dr} &= x_1-x_7.
\end{aligned}$$

**Solution 1(a):**

Because $\dfrac{dx(t)}{dt} = \dfrac{dx}{dr}\dfrac{dr}{dt} = \dfrac{dx}{dr}\dfrac{f}{V}$, then the fraction $f/V$ cancels in the equations, resulting in the new system.

**Problem 1(b)**. Formulate the equations in 1(a) in the system form $\frac{d}{dr}\vec{u} = A\vec{u}$.

**Answer**:

$$A = \begin{pmatrix} -6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

**Problem 1(c)**. Find the eigenvalues of $A$.

**Answer**: $\lambda = 0, -1, -1, -1, -1, -1, -7$

**Solution 1(c)**.

Let $D = |A - \lambda I|$. Replace $-1 - \lambda$ by symbol $u$. Then

$$D = \begin{vmatrix} -5+u & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & u \end{vmatrix}$$

Add each of rows $2, 3, 4, 5, 6$ to row 1. Then $1+u$ is a common factor of row 1 and the determinant multiply rule implies

$$D = (1+u)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & u \end{vmatrix}$$

Cofactor expansion along the last row, plus induction, gives the answer $D = (u+1)(u-6)u^5 = (-\lambda)(-\lambda-7)(-\lambda-1)^5$ with roots $\lambda = 0, -7, -1, -1, -1, -1, -1$.

**Problem 1(d)**. Find the eigenvectors of $A$.

**Solution 1(d)**.

The root $\lambda = -1$ causes us to solve $(A+I)\vec{v} = \vec{0}$, which has coefficient matrix

$$B = \begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \mathbf{rref}(B) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are 2 lead variables and 5 free variables, hence 5 basis vectors

$$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvector for $\lambda = 0$ has all components equal to 1. This fact is found from the equation $(A - (0)I)\vec{v} = \vec{0}$, which has coefficient matrix $A$.

The eigenvector for $\lambda = -7$ has first component $-6$ and the remaining equal to 1. The task begins with the equation $(A - (-7)I)\vec{v} = \vec{0}$, which has coefficient matrix

$$\begin{pmatrix} 7-6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 7-1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7-1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 7-1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 7-1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 7-1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 7-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

The eigenvectors for $\lambda = 0$ and $\lambda = -7$ are respectively

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

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**Problem 1(e)**. Solve the differential equation $\dfrac{d\vec{u}}{dr} = A\vec{u}$ by the eigenanalysis method.

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## Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method**. The eigenpairs of matrix $A$ are required. The matrix $A$ must be diagonalizable, meaning there are $n$ eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms $f_1, \ldots, f_n$ found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \ldots, \vec{d}_n$ in the linear combination

$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{u}_2 + \cdots + f_n(t)\vec{d}_n$$

are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\cdots|\, \vec{d}_n > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\cdots|\, A^{n-1}\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms $f_1, \ldots, f_n$ and $\vec{u}_0$ is the initial data.

**Solution 1(e).**

The eigenvectors corresponding to $\lambda = 0, -7, -1, -1, -1, -1, -1$ are respectively

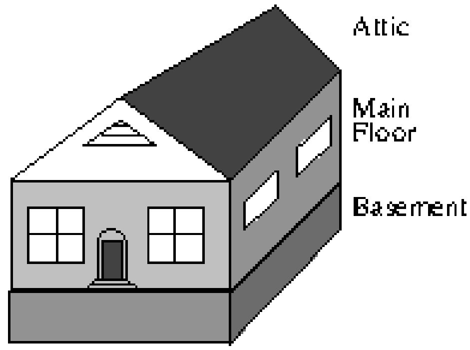
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Eigenanalysis method then implies the solution $\vec{x}(r)$ of $\dfrac{d\vec{x}}{dr} = A\vec{x}$ is given for arbitrary constants $c_1, \ldots, c_7$ by the expression

$$c_1 e^{0r} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-7r} \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_4 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_6 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_7 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

## Problem 2. Home Heating

Consider a typical home with attic, basement and insulated main floor.




### Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

A typical mathematical model is the set of equations

$$\begin{aligned} x' &= \frac{1}{2}(45-x) + \frac{1}{2}(y-x), \\ y' &= \frac{1}{2}(x-y) + \frac{1}{4}(35-y) + \frac{1}{4}(z-y) + 20, \\ z' &= \frac{1}{4}(y-z) + \frac{3}{4}(35-z). \end{aligned}$$

**Problem 2(a)**. Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

**Answer**. $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\quad \vec{b} = \begin{pmatrix} \frac{45}{2} \\ 20 + \frac{35}{4} \\ \frac{105}{4} \end{pmatrix}$, $\quad A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}$

**Solution Details**.

Expand the right side of the system as follows.

$$\begin{aligned} x' &= \frac{45}{2} - \frac{x}{2} + \frac{y}{2} - \frac{x}{2}, \\ y' &= \frac{x}{2} - \frac{y}{2} + \frac{35}{4} - \frac{y}{4} + \frac{z}{4} - \frac{y}{4} + 20 \\ z' &= \frac{y}{4} - \frac{z}{4} + \frac{105}{4} - \frac{3z}{4}. \end{aligned}$$

Then collect on the variables:

$$\begin{aligned} x' &= -x + \frac{y}{2} + \frac{45}{2}, \\ y' &= \frac{x}{2} - y + \frac{z}{4} + 20 + \frac{35}{4} \\ z' &= \frac{y}{4} - z + \frac{105}{4}. \end{aligned}$$

The right side of this system can be written as $A\vec{u} + \vec{b}$. Vector $\vec{b}$ is obtained by formally setting $x = y = z = 0$ on the right. This justifies the answer given.

The matrix $A$ has columns equal to the partial derivatives $\partial_x, \partial_y, \partial_z$ of the right side of the scalar system. This idea is important, because it allows the computation of matrix $A$ **without any of the preceding details**.

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**Problem 2(b)**. The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

**Answer 2(b)**. $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{620}{11} \\ \frac{745}{11} \\ \frac{475}{11} \end{pmatrix} = \begin{pmatrix} 56.36 \\ 67.73 \\ 43.18 \end{pmatrix}$.

**Solution Details**.

The equation upon setting the derivative equal to zero becomes $\vec{0} = A\vec{u} + \vec{b}$ which implies $A\vec{u} = -\vec{b}$ and finally $\vec{u} = -A^{-1}\vec{b}$.

The calculation is done by technology. The maple code:

```
A:=<-1,1/2,0|1/2,-1,1/4|0,1/4,-1>^+; b:=<45/2,20+35/4,105/4>;
-A^(-1).b; evalf(%);
```

The solution can also be obtained by hand from the augmented matrix of $A$ and $-\vec{b}$, using the linear algebra toolkit of swap, combination and multiply.

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**Problem 2(c)**. The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

**Answer 2(c)**: The homogenous scalar general solution is

$$\begin{aligned} x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\ x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\ x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}. \end{aligned}$$

## Three Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method**. Three eigenpairs of matrix $A$ are required. The matrix $A$ must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.
- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors in the linear combination are determined by an explicit formula.

**Solution Details for Problem 2(c)**

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The roots of the characteristic polynomial are used in all three methods. This is the polynomial equation $|A - \lambda I| = 0$, having $n$ roots real and complex, for an $n \times n$ matrix $A$.

Subtract $\lambda$ from the diagonal of $A$ and form the determinant. Then cofactor expansion on row 3 gives

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & -1-\lambda & \frac{1}{4} \\ 0 & \frac{1}{4} & -1-\lambda \end{vmatrix} = (-1-\lambda)\left(-\frac{1}{16} + (-1-\lambda)^2 - \frac{1}{4}\right).$$

The roots are $-1, -a, -b$ where $a = 1 + \sqrt{5}/4 = 1.56, b = 1 - \sqrt{5}/4 = 0.44$. The three roots are distinct, real and negative.

## Eigenanalysis Method

The eigenpairs must be found, in order to assemble the solution vector $\vec{u}(t)$. Technology can be used to find the answers, which are

$$\left(-1, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\right), \quad \left(-a, \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix}\right), \quad \left(-b, \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix}\right).$$

Without technology, there are three homogeneous problems to solve of the form $B\vec{v} = \vec{0}$, for eigenvector $\vec{v}$. Enumerated, they are:

**Case** $\lambda = -1$. Then $B = A + I = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

**Case** $\lambda = -a$. Then $B = A + aI = \begin{pmatrix} \frac{\sqrt{5}}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{5}}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4} \end{pmatrix}$

**Case** $\lambda = -b$. Then $B = A + bI = \begin{pmatrix} -\frac{\sqrt{5}}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{5}}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{\sqrt{5}}{4} \end{pmatrix}$

In each case, the system $B\vec{v} = \vec{0}$ is solved using the last frame algorithm (there is in each case one free variable). The eigenvector reported is the partial derivative of the general solution on the invented symbol $t_1$, which was assigned to the free variable.

### Application of the Theorem

System $\vec{u}' = A\vec{u}$ is solved in the diagonalizable case in terms of the eigenpairs of $A$, denoted as $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The solution of $\vec{u}' = A\vec{u}$ is given by the formula

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3.$$

In the present case, the solution is

$$\vec{u}(t) = c_1 e^{-t}\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-at}\begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix} + c_3 e^{-bt}\begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix}.$$

Symbols $c_1, c_2, c_3$ in the solution are arbitrary constants, uniquely determined by initial conditions. In scalar form, the solution is

$$\begin{aligned}
x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\
x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\
x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}.
\end{aligned}$$

## Laplace Transform Method

The **Laplace Method** for solving $\vec{u}'(t) = A\vec{u}(t)$ is based upon transforming all differential equations into the frequency domain. Then time variable $t$ no longer appears, being replaced by the frequency variable $s$.

The homogeneous system of differential equations is

$$\begin{aligned}
x' &= -x + \frac{y}{2}, \\
y' &= \frac{x}{2} - y + \frac{z}{4} + 20 \\
z' &= \frac{y}{4} - z.
\end{aligned}$$

Transforming to the $s$-domain uses the parts rule $\mathcal{L}(f'(t)) = s\,\mathcal{L}(f(t) - f(0)$. Then

$$\begin{aligned}
s\,\mathcal{L}(x) - x(0) &= -\,\mathcal{L}(x) + \frac{1}{2}\,\mathcal{L}(y), \\
s\,\mathcal{L}(y) - y(0) &= \frac{1}{2}\,\mathcal{L}(x) - \mathcal{L}(y) + \frac{1}{4}\,\mathcal{L}(z) + 20 \\
s\,\mathcal{L}(z) - z(0) &= \frac{1}{4}\,\mathcal{L}(y) - \mathcal{L}(z).
\end{aligned}$$

View these equations as linear algebraic equations for the symbols $\mathcal{L}(x), \mathcal{L}(y), \mathcal{L}(z)$. Move terms left and right to re-write the scalar equations as a matrix system

$$\begin{pmatrix} s+1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & s+1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & s+1 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

The system is solved by inverting the coefficient matrix $C$ on the left, using the adjugate formula $C^{-1} = \textbf{adj}(C)/|C|$. Write the answer as

$$C^{-1}(s) = \begin{pmatrix} s+1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & s+1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & s+1 \end{pmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{pmatrix} s^2+2s+\frac{15}{16} & \frac{s+1}{2} & \frac{1}{8} \\ \frac{s+1}{2} & (s+1)^2 & \frac{s+1}{4} \\ \frac{1}{8} & \frac{s+1}{4} & s^2+2s+\frac{3}{4} \end{pmatrix}.$$

Symbol $\Delta(s) = (s+1)(s+a)(s+b)$ is the determinant of $C(s)$. Then

$$\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = C^{-1}(s) \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

Backward table steps require solving nine equations of the form $\mathcal{L}(f(t)) = \frac{p(s)}{\Delta(s)}$. A computer algebra system reduces the effort, able to write $C^{-1}(s) = \mathcal{L}(\Phi(t))$, using symbols $f_1 = e^{-t}, f_2 = e^{-at}, f_3 = e^{-bt}$, where

$$\Phi(t) = \begin{pmatrix} \frac{1}{5}f_1 + \frac{2}{5}f_2 + \frac{2}{5}f_3 & \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 \\ \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{2}f_2 + \frac{1}{2}f_3 & \frac{1}{2\sqrt{5}}(f_3 - f_2) \\ \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 & \frac{1}{2\sqrt{5}}(f_3 - f_2) & \frac{4}{5}f_1 + \frac{1}{10}f_2 + \frac{1}{10}f_3 \end{pmatrix}$$

Then $\mathcal{L}(\vec{u}(t)) = \mathcal{L}(\Phi(t)\vec{u}(0))$ implies by Lerch's cancelation law the formula

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f_1 + \frac{2}{5}f_2 + \frac{2}{5}f_3 & \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 \\ \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{2}f_2 + \frac{1}{2}f_3 & \frac{1}{2\sqrt{5}}(f_3 - f_2) \\ \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 & \frac{1}{2\sqrt{5}}(f_3 - f_2) & \frac{4}{5}f_1 + \frac{1}{10}f_2 + \frac{1}{10}f_3 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

### The Resolvent Method

The scalar system solved above is exactly

$$(sI - A)\,\mathcal{L}(\vec{u}(t)) = \vec{u}(0), \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}, \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

The system $(sI - A)\,\mathcal{L}(\vec{u}(t)) = \vec{u}(0)$ is called the **resolvent equation**. The inverse of the coefficient matrix, $(sI - A)^{-1}$, is called the **resolvent matrix**, because $\mathcal{L}(\vec{u}(t)) = (sI - A)^{-1}\vec{u}(0)$. If these statements make sense to you, then please use them to solve problems. Otherwise, ignore the information, and solve problems in the same manner as outlined earlier.

### Engineering and Laplace Transforms

Both mechanical engineering and electrical engineering have rich support for Laplace theory. Using Laplace theory has the advantage that many persons can help you through difficult times. Independent persons prefer to choose the method from their own private toolbox.

## Cayley-Hamilton-Ziebur Method

The Ziebur Lemma implies that the solution of the system $\vec{u}'(t) = A\vec{u}(t)$ is given by the formula

$$\vec{u}(t) = \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}.$$

**THEOREM**. Vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are uniquely determined by initial condition $\vec{u}(0)$, which is a column vector of prescribed constants, by the matrix equation.

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left(W(0)^T\right)^{-1}$$

Symbol $W(t)$ is the Wronskian matrix of the three Euler solution atoms. Notation $\langle \vec{A} | \vec{B} | \vec{C} \rangle$ is the augmented matrix of the three columns vectors $\vec{A}, \vec{B}, \vec{C}$.

**Illustration**. For the heating example, with $a = 1 + \sqrt{5}/4 = 1.56, b = 1 - \sqrt{5}/4 = 0.44$, the Euler solution atoms are $e^{-t}, e^{-at}, e^{-bt}$. The Wronskian matrix is

$$W(t) = \begin{pmatrix} e^{-t} & e^{-at} & e^{-bt} \\ -e^{-t} & -ae^{-at} & -be^{-bt} \\ e^{-t} & a^2 e^{-at} & b^2 e^{-bt} \end{pmatrix}, \quad W(0) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}.$$

$$\text{Then } (W(0)^T)^{-1} = \begin{pmatrix} -\frac{11}{5} & \frac{8}{5} - \frac{2}{5}\sqrt{5} & \frac{8}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{32}{5} & \frac{16}{5} - \frac{2}{5}\sqrt{5} & \frac{16}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{16}{5} & \frac{8}{5} & \frac{8}{5} \end{pmatrix}.$$

For initial state $\vec{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle = \begin{pmatrix} 1 & -1 & 5/4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/8 \end{pmatrix}$. Then

$$\vec{u}(t) = \langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle \begin{pmatrix} e^{-t} \\ e^{-at} \\ e^{-bt} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5/4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/8 \end{pmatrix} \begin{pmatrix} -\frac{11}{5} & \frac{8}{5} - \frac{2}{5}\sqrt{5} & \frac{8}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{32}{5} & \frac{16}{5} - \frac{2}{5}\sqrt{5} & \frac{16}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{16}{5} & \frac{8}{5} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} e^{-t} \\ e^{-at} \\ e^{-bt} \end{pmatrix}$$

For general initial state $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$,

$$\langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle = \begin{pmatrix} c_1 & -c_1 + \frac{1}{2}c_2 & \frac{5}{4}c_1 - c_2 + \frac{1}{8}c_3 \\ c_2 & \frac{1}{2}c_1 - c_2 + \frac{1}{4}c_3 & -c_1 + \frac{21}{16}c_2 - \frac{1}{2}c_3 \\ c_3 & \frac{1}{4}c_2 - c_3 & \frac{1}{8}c_1 - \frac{1}{2}c_2 + \frac{17}{16}c_3 \end{pmatrix}.$$

Then $\vec{u}(t)$ is this matrix times $\left(W(0)^T\right)^{-1}$ times the column vector of atoms $e^{-t}, e^{-at}, e^{-bt}$.

**Details for the Theorem**

The idea for finding the three vectors is differentiation of Ziebur's equation, two times, to get three equations

$$\begin{array}{rcrcrcr} \vec{u}(t) & = & \vec{d}_1 e^{-t} & + & \vec{d}_2 e^{-at} & + & \vec{d}_3 e^{-bt}, \\ \vec{u}'(t) & = & -\vec{d}_1 e^{-t} & - & a\vec{d}_2 e^{-at} & - & b\vec{d}_3 e^{-bt}, \\ \vec{u}''(t) & = & \vec{d}_1 e^{-t} & + & a^2\vec{d}_2 e^{-at} & + & b^2\vec{d}_3 e^{-bt}. \end{array}$$

Identities $\vec{u}'(t) = A\vec{u}(t)$ and $\vec{u}''(t) = A\vec{u}'(t) = AA\vec{u}(t)$ imply that the left sides are simplified to

$$\begin{array}{rcrcrcr} \vec{u}(t) & = & \vec{d}_1 e^{-t} & + & \vec{d}_2 e^{-at} & + & \vec{d}_3 e^{-bt}, \\ A\vec{u}(t) & = & -\vec{d}_1 e^{-t} & - & a\vec{d}_2 e^{-at} & - & b\vec{d}_3 e^{-bt}, \\ A^2\vec{u}(t) & = & \vec{d}_1 e^{-t} & + & a^2\vec{d}_2 e^{-at} & + & b^2\vec{d}_3 e^{-bt}. \end{array}$$

The critical idea is to substitute $t = 0$, which because of $e^0 = 1$ gives the following three equations for unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$:

$$\begin{array}{rcrcrcr} \vec{u}(0) & = & \vec{d}_1 & + & \vec{d}_2 & + & \vec{d}_3, \\ A\vec{u}(0) & = & -\vec{d}_1 & - & a\vec{d}_2 & - & b\vec{d}_3, \\ A^2\vec{u}(0) & = & \vec{d}_1 & + & a^2\vec{d}_2 & + & b^2\vec{d}_3. \end{array}$$

How to solve these vector equations for unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$? To begin, solve the scalar system

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where variables $x, y, z$ are the first components of $\vec{d}_1, \vec{d}_2, \vec{d}_3$, and similarly, $b_1, b_2, b_3$ are the first components of vector $\vec{u}(0), A\vec{u}(0), A^2\vec{u}(0)$:

$$\begin{array}{lll} x = \vec{d}_1 \cdot \vec{v}, & b_1 = \vec{u}(0) \cdot \vec{v}, & \\ y = \vec{d}_2 \cdot \vec{v}, & b_2 = A\vec{u}(0) \cdot \vec{v}, & \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \\ z = \vec{d}_3 \cdot \vec{v}, & b_3 = A^2\vec{u}(0) \cdot \vec{v}, & \end{array}$$

The equations also apply to find the second components, using $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then the third components using $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The three systems of equations can be written as one huge matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix} \langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle^T = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle^T$$

Taking the transpose across the equation gives

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}^T = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle$$

Finally, invert the matrix $W(0)^T$ and multiply across the equation on the right to obtain

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left( \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}^T \right)^{-1}$$

This is exactly the equation reported in the theorem,

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left( W(0)^T \right)^{-1}$$

It has been observed that if $f_1 = e^{-t}, f_2 = e^{-at}, f_3 = e^{-bt}$ are replaced by a new basis of solutions such that $W(0) = I$, then $\vec{d}_1 = \vec{u}(0), \vec{d}_2 = A\vec{u}(0), \vec{d}_3 = A^2\vec{u}(0)$. The resulting solution in this case is

$$\vec{u}(t) = f_1(t)\vec{u}(0) + f_2(t)A\vec{u}(0) + f_3(t)A^2\vec{u}(0).$$

## Which Method is the Best?

The eigenanalysis method seems to be the best, because it is a method for simplifying coordinates, hence a shorter answer. Except in the case of complex roots. Or in the case that the matrix $A$ fails to be diagonalizable. In practice, the method used to solve the equation $\vec{u}'(t) = A\vec{u}(t)$ has to be tuned to the expected application. Dynamical systems are an important example. For dynamical systems, the actual solution is less important than its formula, which is a linear combination of Euler solution atoms, according to Cayley-Hamilton-Ziebur.

Laplace theory provides a simple formula for the solution of $\vec{u}'(t) = A\vec{u}(t)$. It has the form

$$\vec{u}(t) = \Phi(t)\vec{u}(0).$$

The matrix $\Phi(t)$ in the Laplace formula is computed from $\mathcal{L}(\Phi(t)) = (sI - A)^{-1}$. Although this computation is nontrivial by hand, computer algebra system automation is possible.

Matrix $\Phi(t)$ is called the **Exponential Matrix**, denoted by $e^{At}$. Computer algebra system `Maple` computes $\Phi(t)$ by the command `LinearAlgebra[MatrixExponential](A,t)`. Both `Matlab` and `Mathematica` support symbolic computation of the matrix exponential $\Phi(t)$.