Math 2250-1 Workout Wednesday Super Quiz Tune-Up (Week 4)

Name and uID: ____________________________________

Write your answer in the space provided. Show work for full credit. You do not need to numerically evaluate all expressions for full credit

1. (10 points) **Problem 1**
   An accelerometer in a smart phone measures an acceleration over $t = 0$ to 1 seconds given by the function

   $$a(t) = -t^3 + 1.5t^2 - 0.5t$$

   measured in cm per seconds squared. The phone starts from rest ($x_0 = 0$, $v_0 = 0$). Write down and solve the differential equations to find the distance traveled after one second.

Solution: The differential equation for velocity is

$$\frac{dv}{dt} = a(t) = -t^3 + 1.5t^2 - 0.5t$$

Integrate to find $v(t)$:

$$v(t) = -0.25t^4 + 0.5t^3 - 0.25t^2 + C$$

Plug in $t = 1$ s:

$$v(0) = 0 \implies C = 0$$

The distance it has traveled at the end of the 1 s is calculated by integrating again to find $x(t)$:

$$x(t) = -0.5t^5 + 0.125t^4 - \frac{0.25}{3}t^3 + C$$

where $C = 0$, so that the distance traveled is

$$x(1) = -0.00833333333$$

2. (10 points) **Problem 2**

Suppose a 10-liter tank contains 9 liters of water and 1 liter of ethanol at time $t = 0$. A 50% concentration solution of water-ethanol is continuously mixed into the tank at a rate $r = 2$ liters per minute. The mixed solution is removed from the tank at the same rate. Let $x(t)$ be the volume in liters of ethanol in the tank at time t in minutes. Find $x(t)$ by solving the appropriate differential equation.

Solution: The mixing tank DE is always of the form

$$\frac{dx}{dt} = r(c_{in} - c_{out})$$

where $r = 2$, $c_{in} = 1/2$, and $c_{out} = x(t)/10$. Plugging in, we get

$$\frac{dx}{dt} = 2(1/2 - x(t)/10) = 1 - \frac{1}{5}x$$

Solving the linear first order DE by means of integrating factor $\rho = e^{\frac{1}{5}t}$:

$$\frac{d}{dt}(xe^{\frac{1}{5}t}) = e^{\frac{1}{5}t}.$$

Then integrate:

$$x(t)e^{\frac{1}{5}t}) = 5e^{\frac{1}{5}t} + C$$

Solve for x:

$$x(t) = 5 + Ce^{-\frac{1}{5}t}$$

Initial condition $x(0) = 1/10$

$$\frac{1}{10} - 5 = C$$

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3. (10 points) **Problem 3**
Consider the differential equation

$$\frac{dx}{dt} = x^2 - 1.$$

(a) Find the equilibrium solutions.

> **Solution:** The equilibrium solutions are the zeros of the right-hand side. Use the quadratic formula
>
> $$(x-1)(x+1) = 0 \implies x = \pm 1$$

(b) Using a phase portrait, decide whether the equilibrium solutions are asymptotically stable or unstable.

> **Solution:** Find the sign of the derivative

Zeros		-1		+1	
$\frac{dx}{dt}$	++	0	−	0	++
$x(t)$	$\nearrow$		$\searrow$		$\nearrow$

> Phase portrait: $\longrightarrow$ -1 $\longleftarrow$ +1 $\longrightarrow$
>
> -1 is asymptotically stable, while +1 is unstable.

(c) Solve the differential equation with $x(0) = 0$.

> **Solution:** This is a separable differential equation
>
> $$\frac{dx}{(x-1)(x+1)} = dt.$$
>
> We use partial fractions.
>
> $$\frac{1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} \quad (1)$$
>
> $$1 = A(x+1) + B(x-1) \quad (2)$$
>
> $$B = -1/2, \quad A = 1/2 \quad (3)$$
>
> Integrate
>
> $$\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = t + C.$$

Rewrite and take the exponential

$$\frac{1}{2}\ln\left|\frac{1-x}{x+1}\right| = t + C \quad \Rightarrow \quad \left|\frac{1-x}{x+1}\right| = Ce^{2t}$$

$$x(0) = 0 \quad \Rightarrow \quad C = 1$$

$$1 - x = (x+1)e^{2t} \quad \Rightarrow \quad x(t) = \frac{1-e^{2t}}{e^{2t}+1}$$

$$x(t) = \frac{-1+e^{-2t}}{e^{-2t}+1}$$