Math 2250 Week 3 Quiz

Name, UID, and section TA: ________________________________

Write your answer in the space provided. Show work for full credit.

1. (3 points) Consider the following differential equation for $x(t)$, which could be modeling a logistic differential equation with harvesting

$$x'(t) = -x^2 + 3x - 2.$$

Find the equilibrium solutions. Then draw the phase diagram and indicate the stability for each equilibrium solution.

Solution: Since $-x^2 + 3x - 2 = -(x^2 - 3x + 2) = -(x-1)(x-2)$, the equilibrium (i.e. constant) solutions are $x \equiv 1, x \equiv 2$. Using the factored form of the DE right side, we see that $x'(t) < 0$ for $x > 2$, $x'(t) > 0$ for $1 < x < 2$, $x'(t) < 0$ for $x < 1$. Therefore the phase diagram is

$$\leftarrow\leftarrow 1 \rightarrow\rightarrow 2 \leftarrow\leftarrow .$$

Thus the solution $x \equiv 1$ is unstable and the solution $x \equiv 2$ is asymptotically stable.

2. (3 points) Compute the partial fractions decomposition for

$$\frac{1}{(x-1)(x-2)}.$$

Solution: We write

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

Equating the numerator on the left with the one on the right yields

$$1 = A(x-2) + B(x-1).$$

For $x = 2$ we must have $1 = B \Rightarrow B = 1$. For $x = 1$ we must have $1 = -A \Rightarrow A = -1$. Thus

$$\frac{1}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{1}{x-2}.$$

"Shortcut:" If we subtract

$$\frac{1}{x-1} - \frac{1}{x-2}$$

and recombine over their common denominator, we can see that the x terms in the numerator will cancel, leaving a constant. If we divide both sides by that constant we will get the partical fractions decomposition:

$$\frac{1}{x-1} - \frac{1}{x-2} = \frac{(x-2)-(x-1)}{(x-1)(x-2)} = \frac{-1}{(x-1)(x-2)}$$

Dividing both sides by -1 yields the same partial fractions decomposition as above.

3. (4 points) Use your work from (2) to solve the initial value problem

$$x'(t) = -x^2 + 3x - 2$$

$$x(0) = 3$$

Solution: Separate variables:

$$x'(t) = -(x-1)(x-2)$$

$$\frac{dx}{(x-1)(x-2)} = -dt$$

Use the partial fractions decomposition and integrate:

$$\int (\frac{-1}{x-1} + \frac{1}{x-2})dx = \int -dt$$

$$ln|\frac{x-2}{x-1}| = -t + C$$

Exponentiate:

$$|\frac{x-2}{x-1}| = e^{-t+C} = Ce^{-t}.$$

So

$$\frac{x-2}{x-1} = Ce^{-t}$$

Substituting $x(0) = 3$ yields $\frac{1}{2} = C$, so

$$\frac{x-2}{x-1} = \frac{1}{2}e^{-t}.$$

Multiply through by $x - 1$ and collect terms to solve for $x(t)$:

$$x - 2 = \frac{1}{2}(x-1)e^{-t} = \frac{1}{2}xe^{-t} - \frac{1}{2}e^{-t} \Rightarrow x(1 - \frac{1}{2}e^{-t}) = 2 - \frac{1}{2}e^{-t}$$

$$\Rightarrow x(t) = \frac{2 - \frac{1}{2}e^{-t}}{1 - \frac{1}{2}e^{-t}}$$