Piecewise Representation of Switching Inputs

- **Unit step.**
- **Pulse function.**
- **Ramp function.**
- **Piecewise-defined functions.**
- **Laplace Pulse Example.**
- **Laplace of a piecewise-defined function.**

Unit Step Function

Definition:

$$\mathbf{u}(t-a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases}$$

Example: Write the piecewise defined function $f(t)$ in terms of unit step functions.

$$f(t) = \begin{cases} \sin t & t \geq \pi, \\ 0 & t < \pi, \end{cases}$$

Solution:

$$\begin{aligned} f(t) &= \sin(t) \begin{cases} 1 & t \geq \pi, \\ 0 & t < \pi. \end{cases} \\ &= \sin(t)\mathbf{u}(t-\pi). \end{aligned}$$

Pulse Function

Definition:

$$\text{pulse}(t, a, b) = \; = \begin{cases} 1 & a \leq t < b, \\ 0 & t < a, t \geq b, \end{cases} \\ \; = \mathbf{u}(t-a) - \mathbf{u}(t-b).$$

Example: Write the piecewise defined function $f(t)$ in terms of pulse functions.

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi, \\ \cos t & \pi \leq t < 2\pi, \\ 0 & \text{else}. \end{cases}$$

Solution:

$$\begin{aligned} f(t) &= \sin(t) \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \text{else} \end{cases} + \cos(t) \begin{cases} 1 & \pi \leq t < 2\pi, \\ 0 & \text{else} \end{cases} \\ &= \sin(t)\,\text{pulse}(t, 0, \pi) + \cos(t)\,\text{pulse}(t, \pi, 2\pi). \end{aligned}$$

Ramp Function

Definition:

$$\begin{aligned}\operatorname{ramp}(t-a) &= \begin{cases} t-a & t \geq a, \\ 0 & t < a, \end{cases} \\ &= (t-a)\mathbf{u}(t-a).\end{aligned}$$

Example: Write the piecewise defined function $f(t)$ in terms of ramp functions.

$$f(t) = \begin{cases} 2t-2 & t \geq 1, \\ 0 & t < 1. \end{cases}$$

Solution:

$$\begin{aligned} f(t) &= (2t-2)\begin{cases} 1 & t \geq 1, \\ 0 & t < 1. \end{cases} \\ &= (2t-2)\mathbf{u}(t-1) \\ &= 2\operatorname{ramp}(t-1). \end{aligned}$$

Piecewise Defined Functions

Definition:

$$f(t) = \begin{cases} f_1(t) & a_1 \leq t < a_2, \\ f_2(t) & a_2 \leq t < a_3, \\ \vdots & \vdots \\ f_n(t) & a_n \leq t < a_{n+1}, \\ 0 & \text{else} \end{cases}$$

The functions $f_1, \ldots, f_n$ must be defined on the whole real line.

Problem: Write the piecewise defined function $f(t)$ in terms of pulse functions.

Solution:

$$\begin{aligned} f(t) &= f_1(t) \begin{cases} 1 & a_1 \leq t < a_2, \\ 0 & \text{else} \end{cases} + \cdots + f_n(t) \begin{cases} 1 & a_n \leq t < a_{n+1}, \\ 0 & \text{else} \end{cases} \\ &= f_1(t)\,\text{pulse}(t, a_1, a_2) + \cdots + f_n(t)\,\text{pulse}(t, a_n, a_{n+1}). \end{aligned}$$

Laplace Pulse Example

$$f(t) = \begin{cases} e^{-t} & 1 \le t < 2, \\ \cos(\pi t) & 2 \le t < 3, \\ 0 & \text{else} \end{cases}$$

Problem: Find $L(f(t))$ by pulse decomposition.

Solution: We use $L(g(t)\mathbf{u}(t,a)) = e^{-as}L(g(t+a))$.

$$\begin{aligned} L(f(t)) &= L(e^{-t}\,\text{pulse}(t,1,2) + \cos(\pi t)\,\text{pulse}(t,2,3)) \\ &= L(e^{-t}\mathbf{u}(t-1) - e^{-t}\mathbf{u}(t-2) + \cos(\pi t)\mathbf{u}(t-2)) - \\ &\quad L(\cos(\pi t)\mathbf{u}(t-3)) \\ &= e^{-s}L(e^{-t-1}) - e^{-2s}L(e^{-t-2}) + e^{-2s}L(\cos(\pi t + 2\pi)) - \\ &\quad e^{-3s}L(\cos(\pi t + 3\pi)) \\ &= \frac{e^{-1-s} - e^{-2-2s}}{s+1} + \frac{se^{-2s} - se^{-3s}}{s^2+\pi^2}. \end{aligned}$$

Laplace of a Piecewise Defined Function

Definition:

$$f(t) = \begin{cases} f_1(t) & a_1 \le t < a_2, \\ f_2(t) & a_2 \le t < a_3, \\ \vdots & \vdots \\ f_n(t) & a_n \le t < a_{n+1}, \\ 0 & \text{else} \end{cases}$$

Assumed is each f_j is defined on the whole real line.

Problem: Find $L(f(t))$ for the piecewise defined function $f(t)$.

Solution: We use $L(g(t)\mathbf{u}(t-a)) = e^{-as}L(g(t+a))$.

$$\begin{aligned} L(f(t)) &= L(f_1(t)\,\text{pulse}(t, a_1, a_2)) + \cdots + L(f_n(t)\,\text{pulse}(t, a_n, a_{n+1})) \\ &= \textstyle\sum_{j=1}^{n} L(f_j(t)\mathbf{u}(t-a_j)) - L(f_j(t)\mathbf{u}(t-a_{j+1})) \\ &= \textstyle\sum_{j=1}^{n} e^{-a_j s}L(f_j(t+a_j)) - e^{-a_{j+1}s}L(f_j(t+a_{j+1})). \end{aligned}$$