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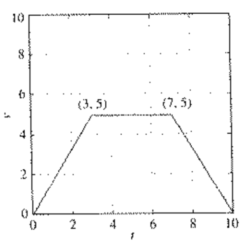

FIGURE 1.2.9. Graph of the velocity function $v(t)$ of Problem 22.

23. What is the maximum height attained by the arrow of part (b) of Example 3?

24. A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

25. The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second per second (m/s^2). How far does the car travel before coming to a stop?

26. A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) its total time in the air.

27. A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

28. A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of the Washington Monument (555 ft high). How long does it take to reach the ground, and with what speed does the baseball strike the ground?

29. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

30. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

31. The skid marks made by an automobile indicated that its brakes were fully applied for a distance of 75 m before it came to a stop. The car in question is known to have a constant deceleration of 20 m/s^2 under these conditions. How fast—in km/h—was the car traveling when the brakes were first applied?

32. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

33. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit?

34. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 33?

35. A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

36. Suppose a woman has enough "spring" in her legs to jump (on earth) from the ground to a height of 2.25 feet. If she jumps straight upward with the same initial velocity on the moon—where the surface gravitational acceleration is (approximately) 5.3 ft/s^2—how high above the surface will she rise?

37. At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

38. At noon a car starts from rest at point A and proceeds with constant acceleration along a straight road toward point C, 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C?

39. If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

40. Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

41. A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exactly 2 seconds after the bomb is released. With what initial velocity should the projectile be fired, in order to hit the bomb at an altitude of exactly 400 feet?

42. A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? (As in Example 2, ignore the moon's gravitational field.)

43. Arthur Clarke's *The Wind from the Sun* (1963) describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.0001g = 0.0098$ m/s^2. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth of the speed $c = 3 \times 10^8$ m/s of light. How long will it take the spacecraft to catch up with the projectile, and how far will it have traveled by then?

44. A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 feet before coming to a stop. But the driver's skid marks at the accident scene measured 210 feet. Assuming the same (constant) deceleration, determine the speed he was actually traveling just prior to the accident.

1.3 Slope Fields and Solution Curves

Consider a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

where the right-hand function $f(x, y)$ involves both the independent variable x and the dependent variable y. We might think of integrating both sides in (1) with respect to x, and hence write $y(x) = \int f(x, y(x))\,dx + C$. However, this approach does not lead to a solution of the differential equation, because the indicated integral involves the *unknown* function $y(x)$ itself, and therefore cannot be evaluated explicitly. Actually, there exists *no* straightforward procedure by which a general differential equation can be solved explicitly. Indeed, the solutions of such a simple-looking differential equation as $y' = x^2 + y^2$ cannot be expressed in terms of the ordinary elementary functions studied in calculus textbooks. Nevertheless, the graphical and numerical methods of this and later sections can be used to construct *approximate* solutions of differential equations that suffice for many practical purposes.

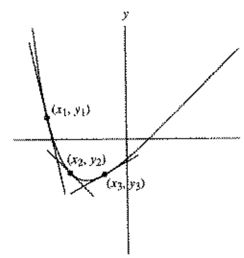

FIGURE 1.3.1. A solution curve for the differential equation $y' = x - y$ together with tangent lines having

- slope $m_1 = x_1 - y_1$ at the point (x_1, y_1);
- slope $m_2 = x_2 - y_2$ at the point (x_2, y_2); and
- slope $m_3 = x_3 - y_3$ at the point (x_3, y_3).

Slope Fields and Graphical Solutions

There is a simple geometric way to think about solutions of a given differential equation $y' = f(x, y)$. At each point (x, y) of the xy-plane, the value of $f(x, y)$ determines a slope $m = f(x, y)$. A solution of the differential equation is simply a differentiable function whose graph $y = y(x)$ has this "correct slope" at each point $(x, y(x))$ through which it passes—that is, $y'(x) = f(x, y(x))$. Thus a **solution curve** of the differential equation $y' = f(x, y)$—the graph of a solution of the equation—is simply a curve in the xy-plane whose tangent line at each point (x, y) has slope $m = f(x, y)$. For instance, Fig. 1.3.1 shows a solution curve of the differential equation $y' = x - y$ together with its tangent lines at three typical points.

This geometric viewpoint suggests a *graphical method* for constructing *approximate* solutions of the differential equation $y' = f(x, y)$. Through each of a representative collection of points (x, y) in the plane we draw a short line segment having the proper slope $m = f(x, y)$. All these line segments constitute a **slope field** (or a **direction field**) for the equation $y' = f(x, y)$.

Example 1 Figures 1.3.2 (a)–(d) show slope fields and solution curves for the differential equation

$$\frac{dy}{dx} = ky \tag{2}$$

with the values $k = 2, 0.5, -1$, and -3 of the parameter k in Eq. (2). Note that each slope field yields important qualitative information about the set of all solutions

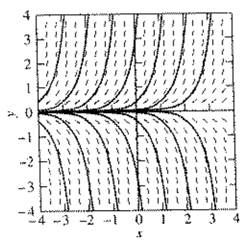

FIGURE 1.3.2(a) Slope field and solution curves for $y' = 2y$.

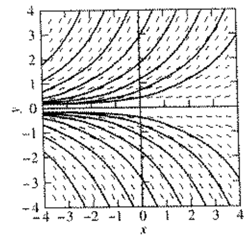

FIGURE 1.3.2(b) Slope field and solution curves for $y' = (0.5)y$.

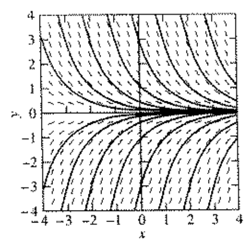

FIGURE 1.3.2(c) Slope field and solution curves for $y' = -y$.

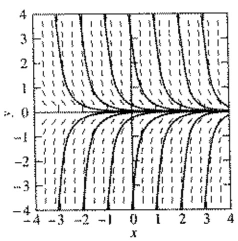

FIGURE 1.3.2(d) Slope field and solution curves for $y' = -3y$.

of the differential equation. For instance, Figs. 1.3.2(a) and (b) suggest that each solution $y(x)$ approaches $\pm\infty$ as $x \to +\infty$ if $k > 0$, whereas Figs. 1.3.2(c) and (d) suggest that $y(x) \to 0$ as $x \to +\infty$ if $k < 0$. Moreover, although the sign of k determines the *direction* of increase or decrease of $y(x)$, its absolute value $|k|$ appears to determine the *rate of change* of $y(x)$. All this is apparent from slope fields like those in Fig. 1.3.2, even without knowing that the general solution of Eq. (2) is given explicitly by $y(x) = Ce^{kx}$. ■

A slope field suggests visually the general shapes of solution curves of the differential equation. Through each point a solution curve should proceed in such a direction that its tangent line is nearly parallel to the nearby line segments of the slope field. Starting at any initial point (a, b), we can attempt to sketch freehand an approximate solution curve that threads its way through the slope field, following the visible line segments as closely as possible.

Example 2 Construct a slope field for the differential equation $y' = x - y$ and use it to sketch an approximate solution curve that passes through the point $(-4, 4)$.

Solution Figure 1.3.3 shows a table of slopes for the given equation. The numerical slope $m = x - y$ appears at the intersection of the horizontal x-row and the vertical y-column of the table. If you inspect the pattern of upper-left to lower-right diagonals in this table, you can see that it was easily and quickly constructed. (Of

$x \backslash y$	−4	−3	−2	−1	0	1	2	3	4
−4	0	−1	−2	−3	−4	−5	−6	−7	−8
−3	1	0	−1	−2	−3	−4	−5	−6	−7
−2	2	1	0	−1	−2	−3	−4	−5	−6
−1	3	2	1	0	−1	−2	−3	−4	−5
0	4	3	2	1	0	−1	−2	−3	−4
1	5	4	3	2	1	0	−1	−2	−3
2	6	5	4	3	2	1	0	−1	−2
3	7	6	5	4	3	2	1	0	−1
4	8	7	6	5	4	3	2	1	0

FIGURE 1.3.3. Values of the slope $y' = x - y$ for $-4 \leqq x, y \leqq 4$.

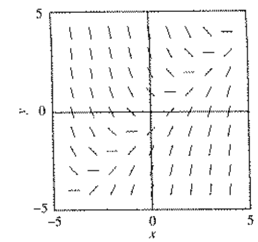

FIGURE 1.3.4. Slope field for $y' = x - y$ corresponding to the table of slopes in Fig. 1.3.3.

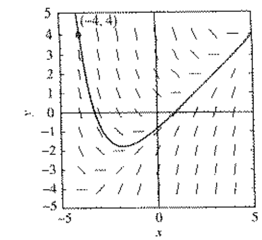

FIGURE 1.3.5. The solution curve through $(-4, 4)$.

course, a more complicated function $f(x, y)$ on the right-hand side of the differential equation would necessitate more complicated calculations.) Figure 1.3.4 shows the corresponding slope field, and Fig. 1.3.5 shows an approximate solution curve sketched through the point $(-4, 4)$ so as to follow this slope field as closely as possible. At each point it appears to proceed in the direction indicated by the nearby line segments of the slope field. ■

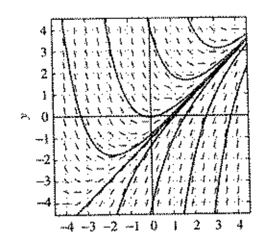

FIGURE 1.3.6. Slope field and typical solution curves for $y' = x - y$.

Although a spreadsheet program (for instance) readily constructs a table of slopes as in Fig. 1.3.3, it can be quite tedious to plot by hand a sufficient number of slope segments as in Fig. 1.3.4. However, most computer algebra systems include commands for quick and ready construction of slope fields with as many line segments as desired; such commands are illustrated in the application material for this section. The more line segments are constructed, the more accurately solution curves can be visualized and sketched. Figure 1.3.6 shows a "finer" slope field for the differential equation $y' = x - y$ of Example 2, together with typical solution curves treading through this slope field.

If you look closely at Fig. 1.3.6, you may spot a solution curve that appears to be a straight line! Indeed, you can verify that the linear function $y = x - 1$ is a solution of the equation $y' = x - y$, and it appears likely that the other solution curves approach this straight line as an asymptote as $x \to +\infty$. This inference illustrates the fact that a slope field can suggest tangible information about solutions that is not at all evident from the differential equation itself. Can you, by tracing the appropriate solution curve in this figure, infer that $y(3) \approx 2$ for the solution $y(x)$ of the initial value problem $y' = x - y$, $y(-4) = 4$?

Applications of Slope Fields

The next two examples illustrate the use of slope fields to glean useful information in physical situations that are modeled by differential equations. Example 3 is based on the fact that a baseball moving through the air at a moderate speed v (less than about 300 ft/s) encounters air resistance that is approximately proportional to v. If the baseball is thrown straight downward from the top of a tall building or from a hovering helicopter, then it experiences both the downward acceleration of gravity and an upward acceleration of air resistance. If the y-axis is directed *downward*, then the ball's velocity $v = dy/dt$ and its gravitational acceleration $g = 32$ ft/s^2 are both positive, while its acceleration due to air resistance is negative. Hence its total acceleration is of the form

$$\frac{dv}{dt} = g - kv. \tag{3}$$

A typical value of the air resistance proportionality constant might be $k = 0.16$.

Example 3 Suppose you throw a baseball straight downward from a helicopter hovering at an altitude of 3000 feet. You wonder whether someone standing on the ground below could conceivably catch it. In order to estimate the speed with which the ball will land, you can use your laptop's computer algebra system to construct a slope field for the differential equation

$$\frac{dv}{dt} = 32 - 0.16v. \tag{4}$$

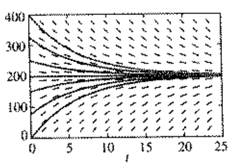

FIGURE 1.3.7. Slope field and typical solution curves for $v' = 32 - 0.16v$.

The result is shown in Fig. 1.3.7, together with a number of solution curves corresponding to different values of the initial velocity $v(0)$ with which you might throw the baseball downward. Note that all these solution curves appear to approach the horizontal line $v = 200$ as an asymptote. This implies that—however you throw it—the baseball should approach the *limiting velocity* $v = 200$ ft/s instead of accelerating indefinitely (as it would in the absence of any air resistance). The handy fact that 60 mi/h = 88 ft/s yields

$$v = 200\frac{\text{ft}}{\text{s}} \times \frac{60 \text{ mi/h}}{88 \text{ ft/s}} \approx 136.36 \frac{\text{mi}}{\text{h}}.$$

Perhaps a catcher accustomed to 100 mi/h fastballs would have some chance of fielding this speeding ball. ■

Comment: If the ball's initial velocity is $v(0) = 200$, then Eq. (4) gives $v'(0) = 32 - (0.16)(200) = 0$, so the ball experiences *no* initial acceleration. Its velocity therefore remains unchanged, and hence $v(t) \equiv 200$ is a constant "equilibrium solution" of the differential equation. If the initial velocity is greater than 200, then the initial acceleration given by Eq. (4) is negative, so the ball slows down as it falls. But if the initial velocity is less than 200, then the initial acceleration given by (4) is positive, so the ball speeds up as it falls. It therefore seems quite reasonable that, because of air resistance, the baseball will approach a limiting velocity of 200 ft/s—whatever initial velocity it starts with. You might like to verify that—in the absence of air resistance—this ball would hit the ground at over 300 mi/h. ■

In Section 2.1 we will discuss in detail the logistic differential equation

$$\frac{dP}{dt} = kP(M - P) \tag{5}$$

that often is used to model a population $P(t)$ that inhabits an environment with *carrying capacity* M. This means that M is the maximum population that this environment can sustain on a long-term basis (in terms of the maximum available food, for instance).

Example 4 If we take $k = 0.0004$ and $M = 150$, then the logistic equation in (5) takes the form

$$\frac{dP}{dt} = 0.0004P(150 - P) = 0.06P - 0.0004P^2. \tag{6}$$

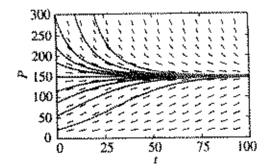

FIGURE 1.3.8. Slope field and typical solution curves for $P' = 0.06P - 0.0004P^2$.

The positive term $0.06P$ on the right in (6) corresponds to natural growth at a 6% annual rate (with time t measured in years). The negative term $-0.0004P^2$ represents the inhibition of growth due to limited resources in the environment.

Figure 1.3.8 shows a slope field for Eq. (6), together with a number of solution curves corresponding to possible different values of the initial population $P(0)$. Note that all these solution curves appear to approach the horizontal line $P = 150$ as an asymptote. This implies that—whatever the initial population—the population $P(t)$ approaches the *limiting population* $P = 150$ as $t \to \infty$. ■

Comment: If the initial population is $P(0) = 150$, then Eq. (6) gives

$$P'(0) = 0.0004(150)(150 - 150) = 0,$$

so the population experiences *no* initial (instantaneous) change. It therefore remains unchanged, and hence $P(t) \equiv 150$ is a constant "equilibrium solution" of the differential equation. If the initial population is greater than 150, then the initial rate of change given by (6) is negative, so the population immediately begins to decrease. But if the initial population is less than 150, then the initial rate of change given by (6) is positive, so the population immediately begins to increase. It therefore seems quite reasonable to conclude that the population will approach a limiting value of 150—whatever the (positive) initial population. ■

Existence and Uniqueness of Solutions

Before one spends much time attempting to solve a given differential equation, it is wise to know that solutions actually *exist*. We may also want to know whether there is only one solution of the equation satisfying a given initial condition—that is, whether its solutions are *unique*.

Example 5 **(a)** [Failure of existence] The initial value problem

$$y' = \frac{1}{x}, \quad y(0) = 0 \tag{7}$$

has *no* solution, because no solution $y(x) = \int (1/x)\,dx = \ln|x| + C$ of the differential equation is defined at $x = 0$. We see this graphically in Fig. 1.3.9, which shows a direction field and some typical solution curves for the equation $y' = 1/x$. It is apparent that the indicated direction field "forces" all solution curves near the y-axis to plunge downward so that none can pass through the point $(0, 0)$.

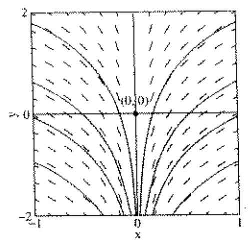

FIGURE 1.3.9. Direction field and typical solution curves for the equation $y' = 1/x$.

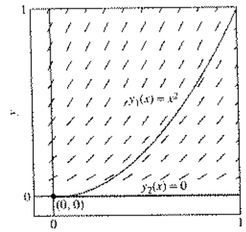

FIGURE 1.3.10. Direction field and two different solution curves for the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$.

(b) [Failure of uniqueness] On the other hand, you can readily verify that the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0 \tag{8}$$

has the *two* different solutions $y_1(x) = x^2$ and $y_2(x) \equiv 0$ (see Problem 27). Figure 1.3.10 shows a direction field and these two different solution curves for the initial value problem in (8). We see that the curve $y_1(x) = x^2$ threads its way through the indicated direction field, whereas the differential equation $y' = 2\sqrt{y}$ specifies slope $y' = 0$ along the x-axis $y_2(x) = 0$. ■

Example 5 illustrates the fact that, before we can speak of "the" solution of an initial value problem, we need to know that it has *one and only one* solution. Questions of existence and uniqueness of solutions also bear on the process of mathematical modeling. Suppose that we are studying a physical system whose behavior is completely determined by certain initial conditions, but that our proposed mathematical model involves a differential equation *not* having a unique solution satisfying those conditions. This raises an immediate question as to whether the mathematical model adequately represents the physical system.

The theorem stated below implies that the initial value problem $y' = f(x, y)$, $y(a) = b$ has one and only one solution defined near the point $x = a$ on the x-axis, provided that both the function f and its partial derivative $\partial f/\partial y$ are continuous near the point (a, b) in the xy-plane. Methods of proving existence and uniqueness theorems are discussed in Appendix A.

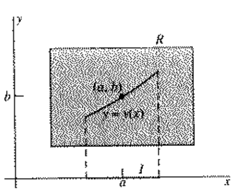

FIGURE 1.3.11. The rectangle R and x-interval I of Theorem 1, and the solution curve $y = y(x)$ through the point (a, b).

THEOREM 1 Existence and Uniqueness of Solutions

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b \tag{9}$$

has one and only one solution that is defined on the interval I. (As illustrated in Fig. 1.3.11, the solution interval I may not be as "wide" as the original rectangle R of continuity; see Remark 3 on the next page.)

Remark 1: In the case of the differential equation $dy/dx = -y$ of Example 1 and Fig. 1.3.2(c), both the function $f(x, y) = -y$ and the partial derivative $\partial f/\partial y = -1$ are continuous everywhere, so Theorem 1 implies the existence of a unique solution for any initial data (a, b). Although the theorem ensures existence only on some open interval containing $x = a$, each solution $y(x) = Ce^{-x}$ actually is defined for all x.

Remark 2: In the case of the differential equation $dy/dx = 2\sqrt{y}$ of Example 5(b) and Eq. (8), the function $f(x, y) = 2\sqrt{y}$ is continuous wherever $y > 0$, but the partial derivative $\partial f/\partial y = 1/\sqrt{y}$ is discontinuous when $y = 0$, and hence at the point $(0, 0)$. This is why it is possible for there to exist two different solutions $y_1(x) = x^2$ and $y_2(x) \equiv 0$, each of which satisfies the initial condition $y(0) = 0$.

Remark 3: In Example 7 of Section 1.1 we examined the especially simple differential equation $dy/dx = y^2$. Here we have $f(x, y) = y^2$ and $\partial f/\partial y = 2y$. Both of these functions are continuous everywhere in the xy-plane, and in particular on the rectangle $-2 < x < 2$, $0 < y < 2$. Because the point $(0, 1)$ lies in the interior of this rectangle, Theorem 1 guarantees a unique solution—necessarily a continuous function—of the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(0) = 1 \tag{10}$$

on some open x-interval containing $a = 0$. Indeed this is the solution

$$y(x) = \frac{1}{1 - x}$$

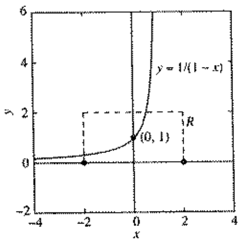

FIGURE 1.3.12. The solution curve through the initial point $(0, 1)$ leaves the rectangle R before it reaches the right side of R.

that we discussed in Example 7. But $y(x) = 1/(1 - x)$ is discontinuous at $x = 1$, so our unique continuous solution does not exist on the entire interval $-2 < x < 2$. Thus the solution interval I of Theorem 1 may not be as wide as the rectangle R where f and $\partial f/\partial y$ are continuous. Geometrically, the reason is that the solution curve provided by the theorem may leave the rectangle—wherein solutions of the differential equation are guaranteed to exist—before it reaches the one or both ends of the interval (see Fig. 1.3.12). ■

The following example shows that, if the function $f(x, y)$ and/or its partial derivative $\partial f/\partial y$ fail to satisfy the continuity hypothesis of Theorem 1, then the initial value problem in (9) may have *either* no solution *or* many—even infinitely many—solutions.

Example 6 Consider the first-order differential equation

$$x\frac{dy}{dx} = 2y. \tag{11}$$

Applying Theorem 1 with $f(x, y) = 2y/x$ and $\partial f/\partial y = 2/x$, we conclude that Eq. (11) must have a unique solution near any point in the xy-plane where $x \neq 0$. Indeed, we see immediately by substitution in (11) that

$$y(x) = Cx^2 \tag{12}$$

satisfies Eq. (11) for any value of the constant C and for all values of the variable x. In particular, the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = 0 \tag{13}$$

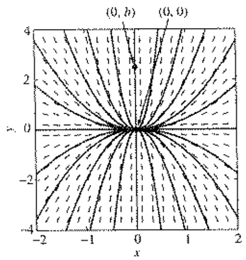

FIGURE 1.3.13. There are infinitely many solution curves through the point (0, 0), but no solution curves through the point (0, b) if $b \neq 0$.

has infinitely many different solutions, whose solution curves are the parabolas $y = Cx^2$ illustrated in Fig. 1.3.13. (In case $C = 0$, the "parabola" is actually the x-axis $y = 0$.)

Observe that all these parabolas pass through the origin (0, 0), but none of them passes through any other point on the y-axis. It follows that the initial value problem in (13) has infinitely many solutions, but the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = b \tag{14}$$

has no solution if $b \neq 0$.

Finally, note that through any point off the y-axis there passes only one of the parabolas $y = Cx^2$. Hence, if $a \neq 0$, then the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(a) = b \tag{15}$$

has a unique solution on any interval that contains the point $x = a$ but not the origin $x = 0$. In summary, the initial value problem in (15) has

- a unique solution near (a, b) if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many solutions if $a = b = 0$. ∎

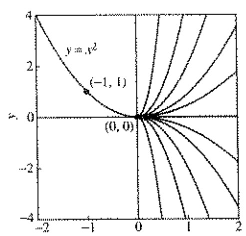

FIGURE 1.3.14. There are infinitely many solution curves through the point (1, −1).

Still more can be said about the initial value problem in (15). Consider a typical initial point off the y-axis—for instance, the point (−1, 1) indicated in Fig. 1.3.14. Then for any value of the constant C the function defined by

$$y(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ Cx^2 & \text{if } x > 0 \end{cases} \tag{16}$$

is continuous and satisfies the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(-1) = 1. \tag{17}$$

For a particular value of C, the solution curve defined by (16) consists of the left half of the parabola $y = x^2$ and the right half of the parabola $y = Cx^2$. Thus the unique solution curve near (−1, 1) branches at the origin into the infinitely many solution curves illustrated in Fig. 1.3.14.

We therefore see that Theorem 1 (if its hypotheses are satisfied) guarantees uniqueness of the solution near the initial point (a, b), but a solution curve through (a, b) may eventually branch elsewhere so that uniqueness is lost. Thus a solution may exist on a larger interval than one on which the solution is unique. For instance, the solution $y(x) = x^2$ of the initial value problem in (17) exists on the whole x-axis, but this solution is unique only on the negative x-axis $-\infty < x < 0$.

1.3 Problems

In Problems 1 through 10, we have provided the slope field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each slope field.

1. $\dfrac{dy}{dx} = -y - \sin x$

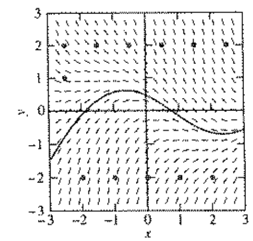

FIGURE 1.3.15

2. $\dfrac{dy}{dx} = x + y$

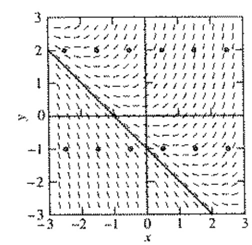

FIGURE 1.3.16

3. $\dfrac{dy}{dx} = y - \sin x$

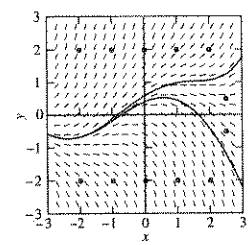

FIGURE 1.3.17

4. $\dfrac{dy}{dx} = x - y$

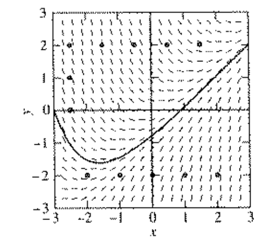

FIGURE 1.3.18

5. $\dfrac{dy}{dx} = y - x + 1$

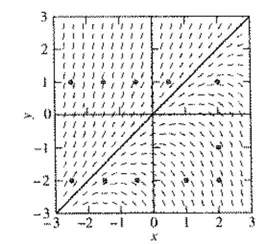

FIGURE 1.3.19

6. $\dfrac{dy}{dx} = x - y + 1$

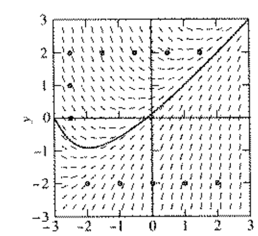

FIGURE 1.3.20

7. $\frac{dy}{dx} = \sin x + \sin y$

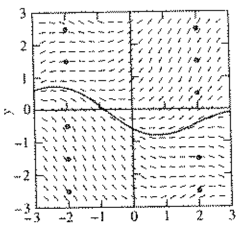

FIGURE 1.3.21

8. $\frac{dy}{dx} = x^2 - y$

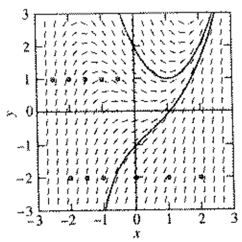

FIGURE 1.3.22

9. $\frac{dy}{dx} = x^2 - y - 2$

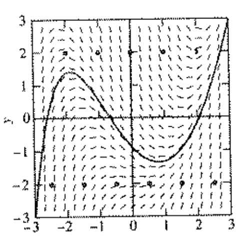

FIGURE 1.3.23

10. $\frac{dy}{dx} = -x^2 + \sin y$

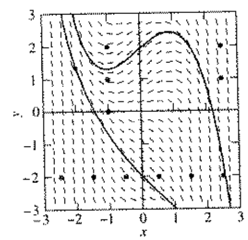

FIGURE 1.3.24

A more detailed version of Theorem 1 says that, if the function $f(x, y)$ is continuous near the point (a, b), then at least one solution of the differential equation $y' = f(x, y)$ exists on some open interval I containing the point $x = a$ and, moreover, that if in addition the partial derivative $\partial f/\partial y$ is continuous near (a, b), then this solution is unique on some (perhaps smaller) interval J. In Problems 11 through 20, determine whether existence of at least one solution of the given initial value problem is thereby guaranteed and, if so, whether uniqueness of that solution is guaranteed.

11. $\frac{dy}{dx} = 2x^2y^2$; $y(1) = -1$

12. $\frac{dy}{dx} = x \ln y$; $y(1) = 1$

13. $\frac{dy}{dx} = \sqrt[3]{y}$; $y(0) = 1$

14. $\frac{dy}{dx} = \sqrt[3]{y}$; $y(0) = 0$

15. $\frac{dy}{dx} = \sqrt{x - y}$; $y(2) = 2$

16. $\frac{dy}{dx} = \sqrt{x - y}$; $y(2) = 1$

17. $y\frac{dy}{dx} = x - 1$; $y(0) = 1$

18. $y\frac{dy}{dx} = x - 1$; $y(1) = 0$

19. $\frac{dy}{dx} = \ln(1 + y^2)$; $y(0) = 0$

20. $\frac{dy}{dx} = x^2 - y^2$; $y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y$, $y(0) = 0$; $y(-4) = ?$

22. $y' = y - x$, $y(4) = 0$; $y(-4) = ?$

Problems 23 and 24 are like Problems 21 and 22, but now use a computer algebra system to plot and print out a slope field for the given differential equation. If you wish (and know how), you can check your manually sketched solution curve by plotting it with the computer.

23. $y' = x^2 + y^2 - 1$, $y(0) = 0$; $y(2) = ?$

24. $y' = x + \frac{1}{2}y^2$, $y(-2) = 0$; $y(2) = ?$

25. You bail out of the helicopter of Example 3 and pull the ripcord of your parachute. Now $k = 1.6$ in Eq. (3), so your downward velocity satisfies the initial value problem

$$\frac{dv}{dt} = 32 - 1.6v, \quad v(0) = 0.$$

In order to investigate your chances of survival, construct a slope field for this differential equation and sketch the appropriate solution curve. What will your limiting velocity be? Will a strategically located haystack do any good? How long will it take you to reach 95% of your limiting velocity?

26. Suppose the deer population $P(t)$ in a small forest satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$$

Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take the number of deer to double? What will be the limiting deer population?

The next seven problems illustrate the fact that, if the hypotheses of Theorem 1 are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.

27. **(a)** Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x - c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x (including the point $x = c$). Construct a figure illustrating the fact that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many different solutions. **(b)** For what values of b does the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ have (i) no solution, (ii) a unique solution that is defined for all x?

28. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y$, $y(a) = b$ has—one, none, or infinitely many.

29. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x - c)^3 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{2/3}$ for all x. Can you also use the "left half" of the cubic $y = (x - c)^3$ in piecing together a solution curve of the differential equation? (See Fig. 1.3.25.) Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.

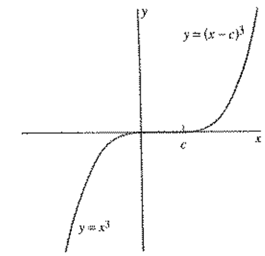

FIGURE 1.3.25. A suggestion for Problem 29.

30. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} +1 & \text{if } x \leqq c, \\ \cos(x - c) & \text{if } c < x < c + \pi, \\ -1 & \text{if } x \geqq c + \pi \end{cases}$$

satisfies the differential equation $y' = -\sqrt{1 - y^2}$ for all x. (Perhaps a preliminary sketch with $c = 0$ will be helpful.) Sketch a variety of such solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = -\sqrt{1 - y^2}$, $y(a) = b$ has.

31. Carry out an investigation similar to that in Problem 30, except with the differential equation $y' = +\sqrt{1 - y^2}$. Does it suffice simply to replace $\cos(x - c)$ with $\sin(x - c)$ in piecing together a solution that is defined for all x?

32. Verify that if $c > 0$, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{if } x^2 \leqq c, \\ (x^2 - c)^2 & \text{if } x^2 > c \end{cases}$$

satisfies the differential equation $y' = 4x\sqrt{y}$ for all x. Sketch a variety of such solution curves for different values of c. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has.

33. If $c \neq 0$, verify that the function defined by $y(x) = x/(cx-1)$ (with graph illustrated in Fig. 1.3.26) satisfies the differential equation $x^2y' + y^2 = 0$ if $x \neq 1/c$. Sketch a variety of such solution curves for different values of c. Also, note the constant-valued function $y(x) \equiv 0$ that does not result from any choice of the constant c. Finally, determine (in terms of a and b) how many different solutions the initial value problem $x^2y' + y^2 = 0$, $y(a) = b$ has.

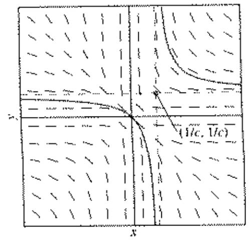

FIGURE 1.3.26. Slope field for $x^2y' + y^2 = 0$ and graph of a solution $y(x) = x/(cx-1)$.

34. (a) Use the direction field of Problem 5 to estimate the values at $x = 1$ of the two solutions of the differential equation $y' = y - x + 1$ with initial values $y(-1) = -1.2$ and $y(-1) = -0.8$.

(b) Use a computer algebra system to estimate the values at $x = 3$ of the two solutions of this differential equation with initial values $y(-3) = -3.01$ and $y(-3) = -2.99$.

The lesson of this problem is that small changes in initial conditions can make big differences in results.

35. (a) Use the direction field of Problem 6 to estimate the values at $x = 2$ of the two solutions of the differential equation $y' = x - y + 1$ with initial values $y(-3) = -0.2$ and $y(-3) = +0.2$.

(b) Use a computer algebra system to estimate the values at $x = 2$ of the two solutions of this differential equation with initial values $y(-3) = -0.5$ and $y(-3) = +0.5$.

The lesson of this problem is that big changes in initial conditions may make only small differences in results.

1.3 Application Computer-Generated Slope Fields and Solution Curves

Widely available computer algebra systems and technical computing environments include facilities to automate the construction of slope fields and solution curves, as do some graphing calculators (see Fig. 1.3.27).

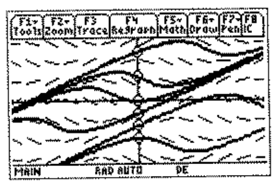

FIGURE 1.3.27. Slope field and solution curves for the differential equation

$$\frac{dy}{dx} = \sin(x - y)$$

with initial points $(0, b)$, $b = -3, -1, -2, 0, 2, 4$ and window $-5 \leq x, y \leq 5$ on a TI-89 graphing calculator.

The applications manual accompanying this textbook includes discussion of *Maple*™, *Mathematica*™, and MATLAB™ resources for the investigation of differential equations. For instance, the *Maple* command

```
with(DEtools):
DEplot(diff(y(x),x)=sin(x-y(x)), y(x), x=-5..5, y=-5..5);
```

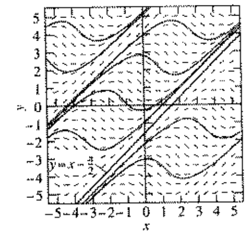

FIGURE 1.3.28. Computer-generated slope field and solution curves for the differential equation $y' = \sin(x - y)$.

and the *Mathematica* command

```
<< Graphics\PlotField.m
PlotVectorField[{1, Sin[x-y]}, {x, -5, 5}, {y, -5, 5}]
```

produce slope fields similar to the one shown in Fig. 1.3.28. Figure 1.3.28 itself was generated with the MATLAB program **dfield** [John Polking and David Arnold, *Ordinary Differential Equations Using* MATLAB, 3rd edition, Upper Saddle River, NJ: Prentice Hall, 2004] that is freely available for educational use (**math.rice.edu/~dfield**). When a differential equation is entered in the **dfield** setup menu (Fig. 1.3.29), you can (with mouse button clicks) plot both a slope field and the solution curve (or curves) through any desired point (or points). Another freely available and user-friendly MATLAB-based ODE package with impressive graphical capabilities is **Iode** (**www.math.uiuc.edu/iode**).

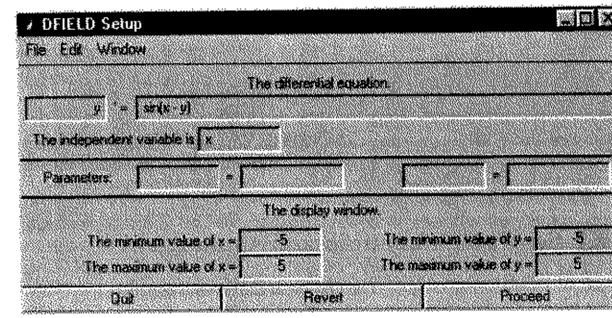

FIGURE 1.3.29. MATLAB **dfield** setup to construct slope field and solution curves for $y' = \sin(x - y)$.

Use a graphing calculator or computer system in the following investigations. You might warm up by generating the slope fields and some solution curves for Problems 1 through 10 in this section.

INVESTIGATION A: Plot a slope field and typical solution curves for the differential equation $dy/dx = \sin(x - y)$, but with a larger window than that of Fig. 1.3.28. With $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, for instance, a number of apparent straight line solution curves should be visible.

(a) Substitute $y = ax + b$ in the differential equation to determine what the coefficients a and b must be in order to get a solution.

(b) A computer algebra system gives the general solution

$$y(x) = x - 2\tan^{-1}\left(\frac{x - 2 - C}{x - C}\right).$$

Plot this solution with selected values of the constant C to compare the resulting solution curves with those indicated in Fig. 1.3.28. Can you see that *no* value of C yields the linear solution $y = x - \pi/2$ corresponding to the initial condition $y(\pi/2) = 0$? Are there any values of C for which the corresponding solution curves lie close to this straight line solution curve?

INVESTIGATION B: For your own personal investigation, let n be the *smallest* digit in your student ID number that is greater than 1, and consider the differential equation

$$\frac{dy}{dx} = \frac{1}{n}\cos(x - ny).$$

(a) First investigate (as in part (a) of Investigation A) the possibility of straight line solutions.

(b) Then generate a slope field for this differential equation, with the viewing window chosen so that you can picture some of these straight lines, plus a sufficient number of nonlinear solution curves that you can formulate a conjecture about what happens to $y(x)$ as $x \to +\infty$. State your inference as plainly as you can. Given the initial value $y(0) = y_0$, try to predict (perhaps in terms of y_0) how $y(x)$ behaves as $x \to +\infty$.

(c) A computer algebra system gives the general solution

$$y(x) = \frac{1}{n}\left[x + 2\tan^{-1}\left(\frac{1}{x - C}\right)\right].$$

Can you make a connection between this symbolic solution and your graphically generated solution curves (straight lines or otherwise)?

1.4 Separable Equations and Applications

The first-order differential equation

$$\frac{dy}{dx} = H(x, y) \tag{1}$$

is called **separable** provided that $H(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = g(x)h(y) = \frac{g(x)}{f(y)},$$

where $h(y) = 1/f(y)$. In this case the variables x and y can be *separated*—isolated on opposite sides of an equation—by writing informally the equation

$$f(y)\,dy = g(x)\,dx,$$

which we understand to be concise notation for the differential equation

$$f(y)\frac{dy}{dx} = g(x). \tag{2}$$

It is easy to solve this special type of differential equation simply by integrating both sides with respect to x:

$$\int f(y(x))\frac{dy}{dx}\,dx = \int g(x)\,dx + C;$$

equivalently,

$$\int f(y)\,dy = \int g(x)\,dx + C. \tag{3}$$

All that is required is that the antiderivatives

$$F(y) = \int f(y)\,dy \quad \text{and} \quad G(x) = \int g(x)\,dx$$

can be found. To see that Eqs. (2) and (3) are equivalent, note the following consequence of the chain rule:

$$D_x[F(y(x))] = F'(y(x))y'(x) = f(y)\frac{dy}{dx} = g(x) = D_x[G(x)],$$

which in turn is equivalent to

$$F(y(x)) = G(x) + C, \tag{4}$$

because two functions have the same derivative on an interval if and only if they differ by a constant on that interval.

Example 1 Solve the initial value problem

$$\frac{dy}{dx} = -6xy, \quad y(0) = 7.$$

Solution Informally, we divide both sides of the differential equation by y and multiply each side by dx to get

$$\frac{dy}{y} = -6x\,dx.$$

Hence

$$\int \frac{dy}{y} = \int (-6x)\,dx;$$

$$\ln|y| = -3x^2 + C.$$

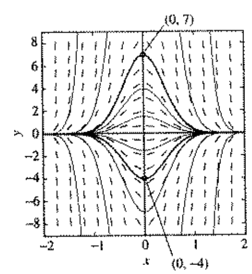

FIGURE 1.4.1. Slope field and solution curves for $y' = -6xy$ in Example 1.

We see from the initial condition $y(0) = 7$ that $y(x)$ is positive near $x = 0$, so we may delete the absolute value symbols:

$$\ln y = -3x^2 + C,$$

and hence

$$y(x) = e^{-3x^2+C} = e^{-3x^2}e^C = Ae^{-3x^2},$$

where $A = e^C$. The condition $y(0) = 7$ yields $A = 7$, so the desired solution is

$$y(x) = 7e^{-3x^2}.$$

This is the upper emphasized solution curve shown in Fig. 1.4.1. ■

Remark: Suppose, instead, that the initial condition in Example 1 had been $y(0) = -4$. Then it would follow that $y(x)$ is *negative* near $x = 0$. We should therefore replace $|y|$ with $-y$ in the integrated equation $\ln|y| = -3x^2 + C$ to obtain

$$\ln(-y) = -3x^2 + C.$$

The initial condition then yields $C = \ln 4$, so $\ln(-y) = -3x^2 + \ln 4$, and hence

$$y(x) = -4e^{-3x^2}.$$

This is the lower emphasized solution curve in Fig. 1.4.1. ■

Example 2 Solve the differential equation

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}. \tag{5}$$

Solution When we separate the variables and integrate both sides, we get

$$\int (3y^2 - 5)\,dy = \int (4-2x)\,dx;$$

$$y^3 - 5y = 4x - x^2 + C. \tag{6}$$

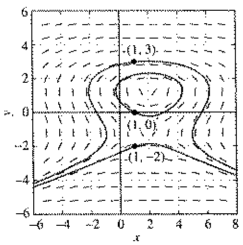

FIGURE 1.4.2. Slope field and solution curves for $y' = (4-2x)/(3y^2-5)$ in Example 2.

This equation is not readily solved for y as an explicit function of x. ■

As Example 2 illustrates, it may or may not be possible or practical to solve Eq. (4) explicitly for y in terms of x. If not, then we call (4) an *implicit solution* of the differential equation in (2). Thus Eq. (6) gives an implicit solution of the differential equation in (5). Although it is not convenient to solve Eq. (6) explicitly in terms of x, we see that each solution curve $y = y(x)$ lies on a contour (or level) curve where the function

$$H(x, y) = x^2 - 4x + y^3 - 5y$$

is constant. Figure 1.4.2 shows several of these contour curves.

Example 3 To solve the initial value problem

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}, \quad y(1) = 3, \tag{7}$$

we substitute $x = 1$ and $y = 3$ in Eq. (6) and get $C = 9$. Thus the desired particular solution $y(x)$ is defined implicitly by the equation

$$y^3 - 5y = 4x - x^2 + 9. \tag{8}$$

The corresponding solution curve $y = y(x)$ lies on the upper contour curve in Fig. 1.4.2—the one passing through $(1, 3)$. Because the graph of a differentiable solution cannot have a vertical tangent line anywhere, it appears from the figure that this particular solution is defined on the interval $(-1, 5)$ but not on the interval $(-3, 7)$. ■

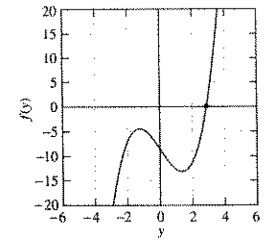

FIGURE 1.4.3. Graph of $f(y) = y^3 - 5y - 9$.

Remark 1: When a specific value of x is substituted in Eq. (8), we can attempt to solve numerically for y. For instance, $x = 4$ yields the equation

$$f(y) = y^3 - 5y - 9 = 0.$$

Figure 1.4.3 shows the graph of f. With a graphing calculator we can solve for the single real root $y \approx 2.8552$. This yields the value $y(4) \approx 2.8552$ of the particular solution in Example 3.

Remark 2: If the initial condition in (7) is replaced with the condition $y(1) = 0$, then the resulting particular solution of the differential equation in (5) lies on the lower "half" of the oval contour curve in Fig. 1.4.2. It appears that this particular solution through $(1, 0)$ is defined on the interval $(0, 4)$ but not on the interval $(-1, 5)$. On the other hand, with the initial condition $y(1) = -2$ we get the lower contour curve in Fig. 1.4.2. This particular solution is defined for all x. Thus the initial condition can determine whether a particular solution is defined on the whole real line or only on some bounded interval. With a computer algebra system one can readily calculate a table of values of the y-solutions of Eq. (8) for x-values at desired increments from $x = -1$ to $x = 5$ (for instance). Such a table of values serves effectively as a "numerical solution" of the initial value problem in (7). ■

Implicit, General, and Singular Solutions

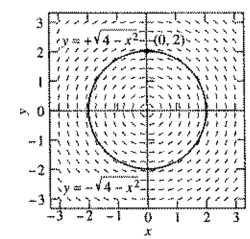

FIGURE 1.4.4. Slope field and solution curves for $y' = -x/y$.

The equation $K(x, y) = 0$ is commonly called an **implicit solution** of a differential equation if it is satisfied (on some interval) by some solution $y = y(x)$ of the differential equation. But note that a particular solution $y = y(x)$ of $K(x, y) = 0$ may or may not satisfy a given initial condition. For example, differentiation of $x^2 + y^2 = 4$ yields

$$x + y\frac{dy}{dx} = 0,$$

so $x^2 + y^2 = 4$ is an implicit solution of the differential equation $x + yy' = 0$. But only the first of the two explicit solutions

$$y(x) = +\sqrt{4-x^2} \quad \text{and} \quad y(x) = -\sqrt{4-x^2}$$

satisfies the initial condition $y(0) = 2$ (Fig. 1.4.4).

Remark 1: You should not assume that every possible algebraic solution $y = y(x)$ of an implicit solution satisfies the same differential equation. For instance, if we multiply the implicit solution $x^2 + y^2 - 4 = 0$ by the factor $(y - 2x)$, then we get the new implicit solution

$$(y - 2x)(x^2 + y^2 - 4) = 0$$

that yields (or "contains") not only the previously noted explicit solutions $y = +\sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$ of the differential equation $x + yy' = 0$, but also the additional function $y = 2x$ that does *not* satisfy this differential equation.

Remark 2: Similarly, solutions of a given differential equation can be either gained or lost when it is multiplied or divided by an algebraic factor. For instance, consider the differential equation

$$(y - 2x)y\frac{dy}{dx} = -x(y - 2x) \tag{9}$$

having the obvious solution $y = 2x$. But if we divide both sides by the common factor $(y - 2x)$, then we get the previously discussed differential equation

$$y\frac{dy}{dx} = -x, \quad \text{or} \quad x + y\frac{dy}{dx} = 0, \tag{10}$$

of which $y = 2x$ is *not* a solution. Thus we "lose" the solution $y = 2x$ of Eq. (9) upon its division by the factor $(y - 2x)$; alternatively, we "gain" this new solution when we multiply Eq. (10) by $(y - 2x)$. Such elementary algebraic operations to simplify a given differential equation before attempting to solve it are common in practice, but the possibility of loss or gain of such "extraneous solutions" should be kept in mind. ■

A solution of a differential equation that contains an "arbitrary constant" (like the constant C in the solution of Examples 1 and 2) is commonly called a **general solution** of the differential equation; any particular choice of a specific value for C yields a single particular solution of the equation.

The argument preceding Example 1 actually suffices to show that *every* particular solution of the differential equation $f(y)y' = g(x)$ in (2) satisfies the equation $F(y(x)) = G(x) + C$ in (4). Consequently, it is appropriate to call (4) not merely ***a*** general solution of (2), but ***the*** general solution of (2).

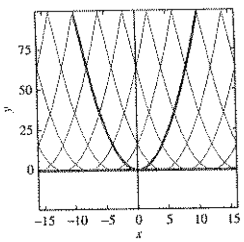

FIGURE 1.4.5. The general solution curves $y = (x - C)^2$ and the singular solution curve $y = 0$ of the differential equation $(y')^2 = 4y$.

In Section 1.5 we shall see that every particular solution of a *linear* first-order differential equation is contained in its general solution. By contrast, it is common for a nonlinear first-order differential equation to have both a general solution involving an arbitrary constant C and one or several particular solutions that cannot be obtained by selecting a value for C. These exceptional solutions are frequently called **singular solutions**. In Problem 30 we ask you to show that the general solution of the differential equation $(y')^2 = 4y$ yields the family of parabolas $y = (x - C)^2$ illustrated in Fig. 1.4.5, and to observe that the constant-valued function $y(x) \equiv 0$ is a singular solution that cannot be obtained from the general solution by any choice of the arbitrary constant C.

Example 4 Find all solutions of the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{2/3}.$$

Solution Separation of variables gives

$$\int \frac{1}{3(y-1)^{2/3}}\,dy = \int 2x\,dx;$$

$$(y-1)^{1/3} = x^2 + C;$$

$$y(x) = 1 + (x^2 + C)^3.$$

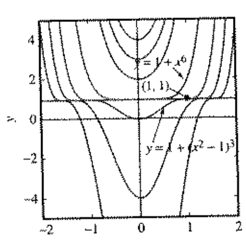

FIGURE 1.4.6. General and singular solution curves for $y' = 6x(y-1)^{2/3}$.

Positive values of the arbitrary constant C give the solution curves in Fig. 1.4.6 that lie above the line $y = 1$, whereas negative values yield those that dip below it. The value $C = 0$ gives the solution $y(x) = 1 + x^6$, but *no* value of C gives the singular solution $y(x) \equiv 1$ that was lost when the variables were separated. Note that the two different solutions $y(x) \equiv 1$ and $y(x) = 1 + (x^2 - 1)^3$ both satisfy the initial condition $y(1) = 1$. Indeed, the whole singular solution curve $y = 1$ consists of points where the solution is not unique and where the function $f(x, y) = 6x(y-1)^{2/3}$ is not differentiable. ■

Natural Growth and Decay

The differential equation

$$\frac{dx}{dt} = kx \quad (k \text{ a constant}) \tag{11}$$

serves as a mathematical model for a remarkably wide range of natural phenomena—any involving a quantity whose time rate of change is proportional to its current size. Here are some examples.

POPULATION GROWTH: Suppose that $P(t)$ is the number of individuals in a population (of humans, or insects, or bacteria) having *constant* birth and death rates β and δ (in births or deaths per individual per unit of time). Then, during a short time interval Δt, approximately $\beta P(t)\,\Delta t$ births and $\delta P(t)\,\Delta t$ deaths occur, so the change in $P(t)$ is given approximately by

$$\Delta P \approx (\beta - \delta)P(t)\,\Delta t,$$

and therefore

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = kP, \tag{12}$$

where $k = \beta - \delta$.

COMPOUND INTEREST: Let $A(t)$ be the number of dollars in a savings account at time t (in years), and suppose that the interest is *compounded continuously* at an annual interest rate r. (Note that 10% annual interest means that $r = 0.10$.) Continuous compounding means that during a short time interval Δt, the amount of interest added to the account is approximately $\Delta A = rA(t)\,\Delta t$, so that

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = rA. \tag{13}$$

RADIOACTIVE DECAY: Consider a sample of material that contains $N(t)$ atoms of a certain radioactive isotope at time t. It has been observed that a constant fraction of those radioactive atoms will spontaneously decay (into atoms of another element or into another isotope of the same element) during each unit of time. Consequently, the sample behaves exactly like a population with a constant death rate and no births. To write a model for $N(t)$, we use Eq. (12) with N in place of P, with $k > 0$ in place of δ, and with $\beta = 0$. We thus get the differential equation

$$\frac{dN}{dt} = -kN. \tag{14}$$

The value of k depends on the particular radioactive isotope.

The key to the method of *radiocarbon dating* is that a constant proportion of the carbon atoms in any living creature is made up of the radioactive isotope ^{14}C of carbon. This proportion remains constant because the fraction of ^{14}C in the atmosphere remains almost constant, and living matter is continuously taking up carbon from the air or is consuming other living matter containing the same constant ratio of ^{14}C atoms to ordinary ^{12}C atoms. This same ratio permeates all life, because organic processes seem to make no distinction between the two isotopes.

The ratio of ^{14}C to normal carbon remains constant in the atmosphere because, although ^{14}C is radioactive and slowly decays, the amount is continuously replenished through the conversion of ^{14}N (ordinary nitrogen) to ^{14}C by cosmic rays bombarding the upper atmosphere. Over the long history of the planet, this decay and replenishment process has come into nearly steady state.

Of course, when a living organism dies, it ceases its metabolism of carbon and the process of radioactive decay begins to deplete its ^{14}C content. There is no replenishment of this ^{14}C, and consequently the ratio of ^{14}C to normal carbon begins to drop. By measuring this ratio, the amount of time elapsed since the death of the organism can be estimated. For such purposes it is necessary to measure the **decay constant** k. For ^{14}C, it is known that $k \approx 0.0001216$ if t is measured in years.

(Matters are not as simple as we have made them appear. In applying the technique of radiocarbon dating, extreme care must be taken to avoid contaminating the sample with organic matter or even with ordinary fresh air. In addition, the cosmic ray levels apparently have not been constant, so the ratio of ^{14}C in the atmosphere has varied over the past centuries. By using independent methods of dating samples, researchers in this area have compiled tables of correction factors to enhance the accuracy of this process.)

DRUG ELIMINATION: In many cases the amount $A(t)$ of a certain drug in the bloodstream, measured by the excess over the natural level of the drug, will decline at a rate proportional to the current excess amount. That is,

$$\frac{dA}{dt} = -\lambda A, \tag{15}$$

where $\lambda > 0$. The parameter λ is called the **elimination constant** of the drug.

The Natural Growth Equation

The prototype differential equation $dx/dt = kx$ with $x(t) > 0$ and k a constant (either negative or positive) is readily solved by separating the variables and integrating:

$$\int \frac{1}{x}\,dx = \int k\,dt;$$

$$\ln x = kt + C.$$

Then we solve for x:

$$e^{\ln x} = e^{kt+C}; \quad x = x(t) = e^C e^{kt} = Ae^{kt}.$$

Because C is a constant, so is $A = e^C$. It is also clear that $A = x(0) = x_0$, so the particular solution of Eq. (11) with the initial condition $x(0) = x_0$ is simply

$$x(t) = x_0 e^{kt}. \tag{16}$$

Because of the presence of the natural exponential function in its solution, the differential equation

$$\frac{dx}{dt} = kx \tag{17}$$

is often called the **exponential** or **natural growth equation.** Figure 1.4.7 shows a typical graph of $x(t)$ in the case $k > 0$; the case $k < 0$ is illustrated in Fig. 1.4.8.

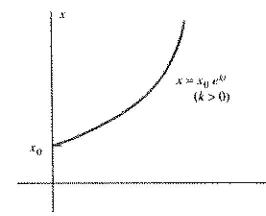

FIGURE 1.4.7. Natural growth.

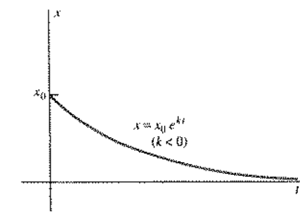

FIGURE 1.4.8. Natural decay.

Example 5 According to data listed at **www.census.gov**, the world's total population reached 6 billion persons in mid-1999, and was then increasing at the rate of about 212 thousand persons each day. Assuming that natural population growth at this rate continues, we want to answer these questions:

(a) What is the annual growth rate k?

(b) What will be the world population at the middle of the 21st century?

(c) How long will it take the world population to increase tenfold—thereby reaching the 60 billion that some demographers believe to be the maximum for which the planet can provide adequate food supplies?

Solution **(a)** We measure the world population $P(t)$ in billions and measure time in years. We take $t = 0$ to correspond to (mid) 1999, so $P_0 = 6$. The fact that P is increasing by 212,000, or 0.000212 billion, persons per day at time $t = 0$ means that

$$P'(0) = (0.000212)(365.25) \approx 0.07743$$

billion per year. From the natural growth equation $P' = kP$ with $t = 0$ we now obtain

$$k = \frac{P'(0)}{P(0)} \approx \frac{0.07743}{6} \approx 0.0129.$$

Thus the world population was growing at the rate of about 1.29% annually in 1999. This value of k gives the world population function

$$P(t) = 6e^{0.0129t}.$$

(b) With $t = 51$ we obtain the prediction

$$P(51) = 6e^{(0.0129)(51)} \approx 11.58 \text{ (billion)}$$

for the world population in mid-2050 (so the population will almost have doubled in the just over a half-century since 1999).

(c) The world population should reach 60 billion when

$$60 = 6e^{0.0129t}; \quad \text{that is, when } t = \frac{\ln 10}{0.0129} \approx 178;$$

and thus in the year 2177. ∎

Note: Actually, the rate of growth of the world population is expected to slow somewhat during the next half-century, and the best current prediction for the 2050 population is "only" 9.1 billion. A simple mathematical model cannot be expected to mirror precisely the complexity of the real world.

The decay constant of a radioactive isotope is often specified in terms of another empirical constant, the *half-life* of the isotope, because this parameter is more convenient. The **half-life** τ of a radioactive isotope is the time required for *half* of it to decay. To find the relationship between k and τ, we set $t = \tau$ and $N = \frac{1}{2}N_0$ in the equation $N(t) = N_0 e^{-kt}$, so that $\frac{1}{2}N_0 = N_0 e^{-k\tau}$. When we solve for τ, we find that

$$\tau = \frac{\ln 2}{k}. \tag{18}$$

For example, the half-life of ^{14}C is $\tau \approx (\ln 2)/(0.0001216)$, approximately 5700 years.

Example 6 A specimen of charcoal found at Stonehenge turns out to contain 63% as much ^{14}C as a sample of present-day charcoal of equal mass. What is the age of the sample?

Solution We take $t = 0$ as the time of the death of the tree from which the Stonehenge charcoal was made and N_0 as the number of ^{14}C atoms that the Stonehenge sample contained then. We are given that $N = (0.63)N_0$ now, so we solve the equation $(0.63)N_0 = N_0 e^{-kt}$ with the value $k = 0.0001216$. Thus we find that

$$t = -\frac{\ln(0.63)}{0.0001216} \approx 3800 \text{ (years)}.$$

Thus the sample is about 3800 years old. If it has any connection with the builders of Stonehenge, our computations suggest that this observatory, monument, or temple—whichever it may be—dates from 1800 B.C. or earlier. ■

Cooling and Heating

According to Newton's law of cooling (Eq. (3) of Section 1.1), the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$. That is,

$$\frac{dT}{dt} = k(A - T), \tag{19}$$

where k is a positive constant. This is an instance of the linear first-order differential equation with constant coefficients:

$$\frac{dx}{dt} = ax + b. \tag{20}$$

It includes the exponential equation as a special case ($b = 0$) and is also easy to solve by separation of variables.

Example 7 A 4-lb roast, initially at 50°F, is placed in a 375°F oven at 5:00 P.M. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125°F. When will the roast be 150°F (medium rare)?

Solution We take time t in minutes, with $t = 0$ corresponding to 5:00 P.M. We also assume (somewhat unrealistically) that at any instant the temperature $T(t)$ of the roast is uniform throughout. We have $T(t) < A = 375$, $T(0) = 50$, and $T(75) = 125$. Hence

$$\frac{dT}{dt} = k(375 - T);$$

$$\int \frac{1}{375 - T}\,dT = \int k\,dt;$$

$$-\ln(375 - T) = kt + C;$$

$$375 - T = Be^{-kt}.$$

Now $T(0) = 50$ implies that $B = 325$, so $T(t) = 375 - 325e^{-kt}$. We also know that $T = 125$ when $t = 75$. Substitution of these values in the preceding equation yields

$$k = -\tfrac{1}{75}\ln\left(\tfrac{250}{325}\right) \approx 0.0035.$$

Hence we finally solve the equation

$$150 = 375 - 325e^{(-0.0035)t}$$

for $t = -[\ln(225/325)]/(0.0035) \approx 105$ (min), the total cooking time required. Because the roast was placed in the oven at 5:00 P.M., it should be removed at about 6:45 P.M. ■

Torricelli's Law

Suppose that a water tank has a hole with area a at its bottom, from which water is leaking. Denote by $y(t)$ the depth of water in the tank at time t, and by $V(t)$ the volume of water in the tank then. It is plausible—and true, under ideal conditions—that the velocity of water exiting through the hole is

$$v = \sqrt{2gy}, \tag{21}$$

which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole (see Problem 35 of Section 1.2). One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the constriction of a water jet from an orifice, $v = c\sqrt{2gy}$, where c is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity we take $c = 1$ in the following discussion.

As a consequence of Eq. (21), we have

$$\frac{dV}{dt} = -av = -a\sqrt{2gy}; \tag{22a}$$

equivalently,

$$\frac{dV}{dt} = -k\sqrt{y} \quad \text{where} \quad k = a\sqrt{2g}. \tag{22b}$$

This is a statement of Torricelli's law for a draining tank. Let $A(y)$ denote the horizontal cross-sectional area of the tank at height y. Then, applied to a thin horizontal slice of water at height $\bar{y}$ with area $A(\bar{y})$ and thickness $d\bar{y}$, the integral calculus method of cross sections gives

$$V(y) = \int_0^y A(\bar{y})\,d\bar{y}.$$

The fundamental theorem of calculus therefore implies that $dV/dy = A(y)$ and hence that

$$\frac{dV}{dt} = \frac{dV}{dy}\cdot\frac{dy}{dt} = A(y)\frac{dy}{dt}. \tag{23}$$

From Eqs. (22) and (23) we finally obtain

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy} = -k\sqrt{y}, \tag{24}$$

an alternative form of Torricelli's law. ■

Example 8 A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

Solution From the right triangle in Fig. 1.4.9, we see that

$$A(y) = \pi r^2 = \pi\left[16 - (4 - y)^2\right] = \pi(8y - y^2).$$

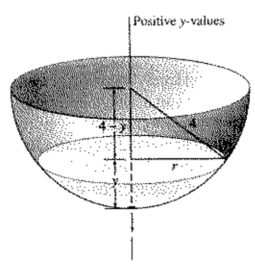

FIGURE 1.4.9. Draining a hemispherical tank.

With $g = 32$ ft/s^2, Eq. (24) becomes

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi\left(\tfrac{1}{24}\right)^2\sqrt{2\cdot 32y}\,;$$

$$\int(8y^{1/2} - y^{3/2})\,dy = -\int \tfrac{1}{72}\,dt;$$

$$\tfrac{16}{3}y^{3/2} - \tfrac{2}{5}y^{5/2} = -\tfrac{1}{72}t + C.$$

Now $y(0) = 4$, so

$$C = \tfrac{16}{3}\cdot 4^{3/2} - \tfrac{2}{5}\cdot 4^{5/2} = \tfrac{448}{15}.$$

The tank is empty when $y = 0$, thus when

$$t = 72\cdot\tfrac{448}{15} \approx 2150\ \text{(s)};$$

that is, about 35 min 50 s. So it takes slightly less than 36 min for the tank to drain. ■

Example 9 In the case of an upright cylindrical tank with constant cross-sectional area A, Torricelli's law in Eq. (24) takes the form

$$\frac{dy}{dt} = -c\sqrt{y}$$

with $c = k/A$. With initial condition $y(0) = 0$ we routinely separate variables and integrate to get

$$y = \tfrac{1}{4}c^2t^2.$$

However, this formula if taken at face value implies that $y > 0$ if $t > 0$. But if the tank is empty at time $t = 0$ as prescribed by the initial condition $y(0) = 0$, then certainly the tank remains empty thereafter, so $y(t) \equiv 0$ for $t > 0$.

To see what is going on here, note that the right-hand side function $f(t, y) = -c\sqrt{y}$ in our differential equation $y' = -c\sqrt{y}$ does *not* satisfy the condition that $\partial f/\partial y$ be continuous at $(0, 0)$, so the existence-uniqueness theorem of Section 1.3 does *not* guarantee uniqueness of a solution near $t = 0$. Indeed, we note the two different but physically meaningful solutions

$$y_1(t) \equiv 0 \quad \text{for all } t, \quad \text{and} \quad y_2(t) = \begin{cases} \tfrac{1}{4}c^2t^2 & \text{for } t < 0, \\ 0 & \text{for } t \geqq 0 \end{cases}$$

of the initial value problem $y' = -c\sqrt{y}$, $y(0) = 0$. The constant solution $y_1(t) \equiv 0$ corresponds to a tank that always has been and always will be empty, while $y_2(t)$ corresponds to a tank draining while $t < 0$ that empties precisely at time $t = 0$ and remains empty thereafter.

Thus this example provides a concrete physical situation described by an initial value problem with non-unique solutions. ■

1.4 Problems

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

1. $\dfrac{dy}{dx} + 2xy = 0$
2. $\dfrac{dy}{dx} + 2xy^2 = 0$
3. $\dfrac{dy}{dx} = y\sin x$
4. $(1 + x)\dfrac{dy}{dx} = 4y$
5. $2\sqrt{x}\,\dfrac{dy}{dx} = \sqrt{1 - y^2}$
6. $\dfrac{dy}{dx} = 3\sqrt{xy}$
7. $\dfrac{dy}{dx} = (64xy)^{1/3}$
8. $\dfrac{dy}{dx} = 2x\sec y$
9. $(1 - x^2)\dfrac{dy}{dx} = 2y$
10. $(1 + x)^2\dfrac{dy}{dx} = (1 + y)^2$
11. $y' = xy^3$
12. $yy' = x(y^2 + 1)$
13. $y^3\dfrac{dy}{dx} = (y^4 + 1)\cos x$
14. $\dfrac{dy}{dx} = \dfrac{1 + \sqrt{x}}{1 + \sqrt{y}}$
15. $\dfrac{dy}{dx} = \dfrac{(x - 1)y^5}{x^2(2y^3 - y)}$
16. $(x^2 + 1)(\tan y)y' = x$
17. $y' = 1 + x + y + xy$ (*Suggestion*: Factor the right-hand side.)
18. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

19. $\dfrac{dy}{dx} = ye^x$, $y(0) = 2e$
20. $\dfrac{dy}{dx} = 3x^2(y^2 + 1)$, $y(0) = 1$
21. $2y\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 - 16}}$, $y(5) = 2$
22. $\dfrac{dy}{dx} = 4x^3y - y$, $y(1) = -3$
23. $\dfrac{dy}{dx} + 1 = 2y$, $y(1) = 1$
24. $(\tan x)\dfrac{dy}{dx} = y$, $y\left(\tfrac{1}{2}\pi\right) = \tfrac{1}{2}\pi$
25. $x\dfrac{dy}{dx} - y = 2x^2y$, $y(1) = 1$
26. $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2$, $y(1) = -1$
27. $\dfrac{dy}{dx} = 6e^{2x-y}$, $y(0) = 0$

28. $2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y, \quad y(4) = \pi/4$

29. **(a)** Find a general solution of the differential equation $dy/dx = y^2$. **(b)** Find a singular solution that is not included in the general solution. **(c)** Inspect a sketch of typical solution curves to determine the points (a, b) for which the initial value problem $y' = y^2$, $y(a) = b$ has a unique solution.

30. Solve the differential equation $(dy/dx)^2 = 4y$ to verify the general solution curves and singular solution curve that are illustrated in Fig. 1.4.5. Then determine the points (a, b) in the plane for which the initial value problem $(y')^2 = 4y$, $y(a) = b$ has **(a)** no solution, **(b)** infinitely many solutions that are defined for all x, **(c)** on some neighborhood of the point $x = a$, only finitely many solutions.

31. Discuss the difference between the differential equations $(dy/dx)^2 = 4y$ and $dy/dx = 2\sqrt{y}$. Do they have the same solution curves? Why or why not? Determine the points (a, b) in the plane for which the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

32. Find a general solution and any singular solutions of the differential equation $dy/dx = y\sqrt{y^2 - 1}$. Determine the points (a, b) in the plane for which the initial value problem $y' = y\sqrt{y^2 - 1}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

33. (Population growth) A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

34. (Population growth) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 h. How long did it take for the population to double?

35. (Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

36. (Radiocarbon dating) Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

37. (Continuously compounded interest) Upon the birth of their first child, a couple deposited \$5000 in an account that pays 8% interest compounded continuously. The interest payments are allowed to accumulate. How much will the account contain on the child's eighteenth birthday?

38. (Continuously compounded interest) Suppose that you discover in your attic an overdue library book on which your grandfather owed a fine of 30 cents 100 years ago. If an overdue fine grows exponentially at a 5% annual rate compounded continuously, how much would you have to pay if you returned the book today?

39. (Drug elimination) Suppose that sodium pentobarbital is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbitol per kilogram of the dog's body weight. Suppose also that sodium pentobarbitol is eliminated exponentially from the dog's bloodstream, with a half-life of 5 h. What single dose should be administered in order to anesthetize a 50-kg dog for 1 h?

40. The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable for human habitation. How long will it be until the region is again habitable? (Ignore the probable presence of other radioactive isotopes.)

41. Suppose that a mineral body formed in an ancient cataclysm—perhaps the formation of the earth itself—originally contained the uranium isotope ^{238}U (which has a half-life of 4.51×10^9 years) but no lead, the end product of the radioactive decay of ^{238}U. If today the ratio of ^{238}U atoms to lead atoms in the mineral body is 0.9, when did the cataclysm occur?

42. A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half-life is about 1.28×10^9 years) and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium?

43. A pitcher of buttermilk initially at 25°C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature of the buttermilk has dropped to 15°C after 20 min. When will it be at 5°C?

44. When sugar is dissolved in water, the amount A that remains undissolved after t minutes satisfies the differential equation $dA/dt = -kA$ $(k > 0)$. If 25% of the sugar dissolves after 1 min, how long does it take for half of the sugar to dissolve?

45. The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential equation $dI/dx = (-1.4)I$. **(a)** At what depth is the intensity half the intensity I_0 at the surface (where $x = 0$)? **(b)** What is the intensity at a depth of 10 m (as a fraction of I_0)? **(c)** At what depth will the intensity be 1% of that at the surface?

46. The barometric pressure p (in inches of mercury) at an altitude x miles above sea level satisfies the initial value problem $dp/dx = (-0.2)p$, $p(0) = 29.92$. **(a)** Calculate the barometric pressure at 10,000 ft and again at 30,000 ft. **(b)** Without prior conditioning, few people can survive when the pressure drops to less than 15 in. of mercury. How high is that?

47. A certain piece of dubious information about phenylethylamine in the drinking water began to spread one day in a city with a population of 100,000. Within a week, 10,000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population of the city has heard the rumor?

48. According to one cosmological theory, there were equal amounts of the two uranium isotopes ^{235}U and ^{238}U at the creation of the universe in the "big bang." At present there are 137.7 atoms of ^{238}U for each atom of ^{235}U. Using the half-lives 4.51×10^9 years for ^{238}U and 7.10×10^8 years for ^{235}U, calculate the age of the universe.

49. A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

50. The amount $A(t)$ of atmospheric pollutants in a certain mountain valley grows naturally and is tripling every 7.5 years.
   - **(a)** If the initial amount is 10 pu (pollutant units), write a formula for $A(t)$ giving the amount (in pu) present after t years.
   - **(b)** What will be the amount (in pu) of pollutants present in the valley atmosphere after 5 years?
   - **(c)** If it will be dangerous to stay in the valley when the amount of pollutants reaches 100 pu, how long will this take?

51. An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (safe units), and 5 months later it is still 10 su.
   - **(a)** Write a formula giving the amount $A(t)$ of radioactive material (in su) remaining after t months.
   - **(b)** What amount of radioactive material will remain after 8 months?
   - **(c)** How long—total number of months or fraction thereof—will it be until $A = 1$ su, so it is safe for people to return to the area?

52. There are now about 3300 different human "language families" in the whole world. Assume that all these are derived from a single original language, and that a language family develops into 1.5 language families every 6 thousand years. About how long ago was the single original human language spoken?

53. Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian "language families." Assume (as in Problem 52) that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

54. A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at time $t = 0$ (hours). After 1 h the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?

55. Suppose that the tank of Problem 54 has a radius of 3 ft and that its bottom hole is circular with radius 1 in. How long will it take the water (initially 9 ft deep) to drain completely?

56. At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 ft high is removed. After 1 h the water in the tank is 9 ft deep. When will the tank be empty?

57. Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \leqq T$ minutes is $V = V_0[1 - (t/T)]^2$.

58. A water tank has the shape obtained by revolving the curve $y = x^{4/3}$ around the y-axis. A plug at the bottom is removed at 12 noon, when the depth of water in the tank is 12 ft. At 1 P.M. the depth of the water is 6 ft. When will the tank be empty?

59. A water tank has the shape obtained by revolving the parabola $x^2 = by$ around the y-axis. The water depth is 4 ft at 12 noon, when a circular plug in the bottom of the tank is removed. At 1 P.M. the depth of the water is 1 ft. **(a)** Find the depth $y(t)$ of water remaining after t hours. **(b)** When will the tank be empty? **(c)** If the initial radius of the top surface of the water is 2 ft, what is the radius of the circular hole in the bottom?

60. A cylindrical tank with length 5 ft and radius 3 ft is situated with its axis horizontal. If a circular bottom hole with a radius of 1 in. is opened and the tank is initially half full of xylene, how long will it take for the liquid to drain completely?

61. A spherical tank of radius 4 ft is full of gasoline when a circular bottom hole with radius 1 in. is opened. How long will be required for all the gasoline to drain from the tank?

62. Suppose that an initially full hemispherical water tank of radius 1 m has its flat side as its bottom. It has a bottom hole of radius 1 cm. If this bottom hole is opened at 1 P.M., when will the tank be empty?

63. Consider the initially full hemispherical water tank of Example 8, except that the radius r of its circular bottom hole is now unknown. At 1 P.M. the bottom hole is opened and at 1:30 P.M. the depth of water in the tank is 2 ft. **(a)** Use Torricelli's law in the form $dV/dt = -(0.6)\pi r^2\sqrt{2gy}$ (taking constriction into account) to determine when the tank will be empty. **(b)** What is the radius of the bottom hole?

64. (The *clepsydra*, or water clock) A 12-h water clock is to be designed with the dimensions shown in Fig. 1.4.10, shaped like the surface obtained by revolving the curve $y = f(x)$ around the y-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the *constant* rate of 4 inches per hour (in./h)?

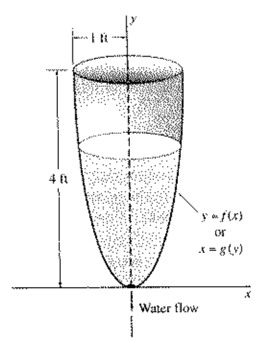

FIGURE 1.4.10. The clepsydra.

65. Just before midday the body of an apparent homicide victim is found in a room that is kept at a constant temperature of 70°F. At 12 noon the temperature of the body is 80°F and at 1 P.M. it is 75°F. Assume that the temperature of the body at the time of death was 98.6°F and that it has cooled in accord with Newton's law. What was the time of death?

66. Early one morning it began to snow at a constant rate. At 7 A.M. a snowplow set off to clear a road. By 8 A.M. it had traveled 2 miles, but it took two more hours (until 10 A.M.) for the snowplow to go an additional 2 miles. (a) Let $t = 0$ when it began to snow and let x denote the distance traveled by the snowplow at time t. Assuming that the snowplow clears snow from the road at a constant rate (in cubic feet per hour, say), show that

$$k\frac{dx}{dt} = \frac{1}{t},$$

where k is a constant. (b) What time did it start snowing? (*Answer*: 6 A.M.)

67. A snowplow sets off at 7 A.M. as in Problem 66. Suppose now that by 8 A.M. it had traveled 4 miles and that by 9 A.M. it had moved an additional 3 miles. What time did it start snowing? This is a more difficult snowplow problem because now a transcendental equation must be solved numerically to find the value of k. (*Answer*: 4:27 A.M.)

68. Figure 1.4.11 shows a bead sliding down a frictionless wire from point P to point Q. The *brachistochrone problem* asks what shape the wire should be in order to minimize the bead's time of descent from P to Q. In June of 1696, John Bernoulli proposed this problem as a public challenge, with a 6-month deadline (later extended to Easter 1697 at George Leibniz's request). Isaac Newton, then retired from academic life and serving as Warden of the Mint in London, received Bernoulli's challenge on January 29, 1697. The very next day he communicated his own solution—the curve of minimal descent time is an arc of an inverted cycloid—to the Royal Society of London. For a modern derivation of this result, suppose the bead starts from rest at the origin P and let $y = y(x)$ be the equation of the desired curve in a coordinate system with the y-axis pointing downward. Then a mechanical analogue of Snell's law in optics implies that

$$\frac{\sin\alpha}{v} = \text{constant}, \qquad \text{(i)}$$

where α denotes the angle of deflection (from the vertical) of the tangent line to the curve—so $\cot\alpha = y'(x)$ (why?)—and $v = \sqrt{2gy}$ is the bead's velocity when it has descended a distance y vertically (from $\text{KE} = \frac{1}{2}mv^2 = mgy = -\text{PE}$).

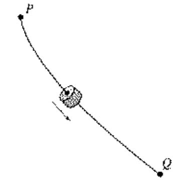

FIGURE 1.4.11. A bead sliding down a wire—the brachistochrone problem.

(a) First derive from Eq. (i) the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}, \qquad \text{(ii)}$$

where a is an appropriate positive constant.

(b) Substitute $y = 2a\sin^2 t$, $dy = 4a\sin t\cos t\,dt$ in (ii) to derive the solution

$$x = a(2t - \sin 2t), \quad y = a(1 - \cos 2t) \qquad \text{(iii)}$$

for which $t = y = 0$ when $x = 0$. Finally, the substitution of $\theta = 2a$ in (iii) yields the standard parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis. [See Example 5 in Section 9.4 of Edwards and Penney, *Calculus: Early Transcendentals*, 7th edition (Upper Saddle River, NJ: Prentice Hall, 2008).]

69. Suppose a uniform flexible cable is suspended between two points $(\pm L, H)$ at equal heights located symmetrically on either side of the x-axis (Fig. 1.4.12). Principles of physics can be used to show that the shape $y = y(x)$ of the hanging cable satisfies the differential equation

$$a\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

where the constant $a = T/\rho$ is the ratio of the cable's tension T at its lowest point $x = 0$ (where $y'(0) = 0$) and its (constant) linear density ρ. If we substitute $v = dy/dx$, $dv/dx = d^2y/dx^2$ in this second-order differential equation, we get the first-order equation

$$a\frac{dv}{dx} = \sqrt{1 + v^2}.$$

Solve this differential equation for $y'(x) = v(x) = \sinh(x/a)$. Then integrate to get the shape function

$$y(x) = a\cosh\left(\frac{x}{a}\right) + C$$

of the hanging cable. This curve is called a *catenary*, from the Latin word for *chain*.

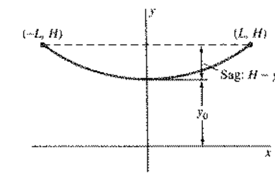

FIGURE 1.4.12. The catenary.

1.4 Application The Logistic Equation

As in Eq. (3) of this section, the solution of a separable differential equation reduces to the evaluation of two indefinite integrals. It is tempting to use a symbolic algebra system for this purpose. We illustrate this approach using the *logistic differential equation*

$$\frac{dx}{dt} = ax - bx^2 \qquad (1)$$

that models a population $x(t)$ with births (per unit time) proportional to x and deaths proportional to x^2. Here we concentrate on the solution of Eq. (1) and defer discussion of population applications to Section 2.1.

If $a = 0.01$ and $b = 0.0001$, for instance, Eq. (1) is

$$\frac{dx}{dt} = (0.01)x - (0.0001)x^2 = \frac{x}{10000}(100 - x). \qquad (2)$$

Separation of variables leads to

$$\int \frac{1}{x(100 - x)}\,dx = \int \frac{1}{10000}\,dt = \frac{t}{10000} + C. \qquad (3)$$

We can evaluate the integral on the left by using the *Maple* command

```
int(1/(x*(100 - x)), x);
```

the *Mathematica* command

```
Integrate[ 1/(x*(100 - x)), x ]
```

or the MATLAB command

```
syms x;   int(1/(x*(100 - x)))
```

Any computer algebra system gives a result of the form

$$\frac{1}{100}\ln x - \frac{1}{100}\ln(x - 100) = \frac{t}{10000} + C \qquad (4)$$

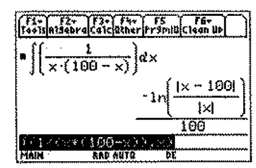

FIGURE 1.4.13. TI-89 screen showing the integral in Eq. (3).

equivalent to the graphing calculator result shown in Fig. 1.4.13.

You can now apply the initial condition $x(0) = x_0$, combine logarithms, and finally exponentiate to solve Eq. (4) for the particular solution

$$x(t) = \frac{100x_0e^{t/100}}{100 - x_0 + x_0e^{t/100}}. \qquad (5)$$

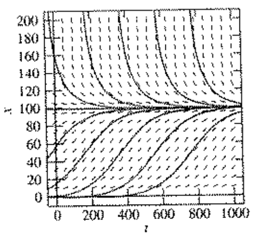

FIGURE 1.4.14. Slope field and solution curves for $x' = (0.01)x - (0.0001)x^2$.

of Eq. (2). The slope field and solution curves shown in Fig. 1.4.14 suggest that, whatever is the initial value x_0, the solution $x(t)$ approaches 100 as $t \to +\infty$. Can you use Eq. (5) to verify this conjecture?

INVESTIGATION: For your own personal logistic equation, take $a = m/n$ and $b = 1/n$ in Eq. (1), with m and n being the *largest* two distinct digits (in either order) in your student ID number.

(a) First generate a slope field for your differential equation and include a sufficient number of solution curves that you can see what happens to the population as $t \to +\infty$. State your inference plainly.

(b) Next use a computer algebra system to solve the differential equation symbolically; then use the symbolic solution to find the limit of $x(t)$ as $t \to +\infty$. Was your graphically based inference correct?

(c) Finally, state and solve a numerical problem using the symbolic solution. For instance, how long does it take x to grow from a selected initial value x_0 to a given target value x_1?

1.5 Linear First-Order Equations

In Section 1.4 we saw how to solve a separable differential equation by integrating *after* multiplying both sides by an appropriate factor. For instance, to solve the equation

$$\frac{dy}{dx} = 2xy \quad (y > 0), \tag{1}$$

we multiply both sides by the factor $1/y$ to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x; \quad \text{that is,} \quad D_x(\ln y) = D_x\left(x^2\right). \tag{2}$$

Because each side of the equation in (2) is recognizable as a *derivative* (with respect to the independent variable x), all that remains are two simple integrations, which yield $\ln y = x^2 + C$. For this reason, the function $\rho(y) = 1/y$ is called an *integrating factor* for the original equation in (1). An **integrating factor** for a differential equation is a function $\rho(x, y)$ such that the multiplication of each side of the differential equation by $\rho(x, y)$ yields an equation in which each side is recognizable as a derivative.

With the aid of the appropriate integrating factor, there is a standard technique for solving the **linear first-order equation**

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3}$$

on an interval on which the coefficient functions $P(x)$ and $Q(x)$ are continuous. We multiply each side in Eq. (3) by the integrating factor

$$\rho(x) = e^{\int P(x)\,dx}. \tag{4}$$

The result is

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)e^{\int P(x)\,dx}y = Q(x)e^{\int P(x)\,dx}. \tag{5}$$

Because

$$D_x\left[\int P(x)\,dx\right] = P(x),$$

the left-hand side is the derivative of the *product* $y(x) \cdot e^{\int P(x)\,dx}$, so Eq. (5) is equivalent to

$$D_x\left[y(x) \cdot e^{\int P(x)\,dx}\right] = Q(x)e^{\int P(x)\,dx}.$$

Integration of both sides of this equation gives

$$y(x)e^{\int P(x)\,dx} = \int \left(Q(x)e^{\int P(x)\,dx}\right) dx + C.$$

Finally, solving for y, we obtain the general solution of the linear first-order equation in (3):

$$y(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right) dx + C\right]. \tag{6}$$

This formula should **not** be memorized. In a specific problem it generally is simpler to use the *method* by which we developed the formula. That is, in order to solve an equation that can be written in the form in Eq. (3) with the coefficient functions $P(x)$ and $Q(x)$ displayed explicitly, you should attempt to carry out the following steps.

METHOD: SOLUTION OF FIRST-ORDER EQUATIONS

1. Begin by calculating the integrating factor $\rho(x) = e^{\int P(x)\,dx}$.
2. Then multiply both sides of the differential equation by $\rho(x)$.
3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:
$$D_x[\rho(x)y(x)] = \rho(x)Q(x).$$
4. Finally, integrate this equation,
$$\rho(x)y(x) = \int \rho(x)Q(x)\,dx + C,$$
then solve for y to obtain the general solution of the original differential equation.

Remark 1: Given an initial condition $y(x_0) = y_0$, you can (as usual) substitute $x = x_0$ and $y = y_0$ into the general solution and solve for the value of C yielding the particular solution that satisfies this initial condition.

Remark 2: You need not supply explicitly a constant of integration when you find the integrating factor $\rho(x)$. For if we replace

$$\int P(x)\,dx \quad \text{with} \quad \int P(x)\,dx + K$$

in Eq. (4), the result is

$$\rho(x) = e^{K + \int P(x)\,dx} = e^K e^{\int P(x)\,dx}.$$

But the constant factor e^K does not affect materially the result of multiplying both sides of the differential equation in (3) by $\rho(x)$, so we might as well take $K = 0$. You may therefore choose for $\int P(x)\,dx$ *any* convenient antiderivative of $P(x)$, without bothering to add a constant of integration. ∎

Example 1 Solve the initial value problem

$$\frac{dy}{dx} - y = \tfrac{11}{8}e^{-x/3}, \quad y(0) = -1.$$

Solution Here we have $P(x) \equiv -1$ and $Q(x) = \frac{11}{8}e^{-x/3}$, so the integrating factor is

$$\rho(x) = e^{\int(-1)\,dx} = e^{-x}.$$

Multiplication of both sides of the given equation by e^{-x} yields

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \tfrac{11}{8}e^{-4x/3}, \tag{7}$$

which we recognize as

$$\frac{d}{dx}\left(e^{-x}y\right) = \tfrac{11}{8}e^{-4x/3}.$$

Hence integration with respect to x gives

$$e^{-x}y = \int \tfrac{11}{8}e^{-4x/3}\,dx = -\tfrac{33}{32}e^{-4x/3} + C,$$

and multiplication by e^x gives the general solution

$$y(x) = Ce^x - \tfrac{33}{32}e^{-x/3}. \tag{8}$$

Substitution of $x = 0$ and $y = -1$ now gives $C = \frac{1}{32}$, so the desired particular solution is

$$y(x) = \tfrac{1}{32}e^x - \tfrac{33}{32}e^{-x/3} = \tfrac{1}{32}\left(e^x - 33e^{-x/3}\right). \quad ∎$$

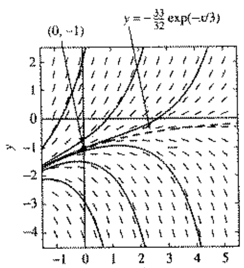

FIGURE 1.5.1. Slope field and solution curves for $y' = y + \frac{11}{8}e^{-x/3}$.

Remark: Figure 1.5.1 shows a slope field and typical solution curves for Eq. (7), including the one passing through the point $(0, -1)$. Note that some solutions grow rapidly in the positive direction as x increases, while others grow rapidly in the negative direction. The behavior of a given solution curve is determined by its initial condition $y(0) = y_0$. The two types of behavior are separated by the particular solution $y(x) = -\frac{33}{32}e^{-x/3}$ for which $C = 0$ in Eq. (8), so $y_0 = -\frac{33}{32}$ for the solution curve that is dashed in Fig. 1.5.1. If $y_0 > -\frac{33}{32}$, then $C > 0$ in Eq. (8), so the term e^x eventually dominates the behavior of $y(x)$, and hence $y(x) \to +\infty$ as $x \to +\infty$. But if $y_0 < -\frac{33}{32}$, then $C < 0$, so both terms in $y(x)$ are negative and therefore $y(x) \to -\infty$ as $x \to +\infty$. Thus the initial condition $y_0 = -\frac{33}{32}$ is *critical* in the sense that solutions that start above $-\frac{33}{32}$ on the y-axis grow in the positive direction, while solutions that start lower than $-\frac{33}{32}$ grow in the negative direction as $x \to +\infty$. The interpretation of a mathematical model often hinges on finding such a critical condition that separates one kind of behavior of a solution from a different kind of behavior. ∎

Example 2 Find a general solution of

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x. \tag{9}$$

Solution After division of both sides of the equation by $x^2 + 1$, we recognize the result

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

as a first-order linear equation with $P(x) = 3x/(x^2+1)$ and $Q(x) = 6x/(x^2+1)$. Multiplication by

$$\rho(x) = \exp\left(\int \frac{3x}{x^2+1}\,dx\right) = \exp\left(\tfrac{3}{2}\ln(x^2+1)\right) = (x^2+1)^{3/2}$$

yields

$$(x^2+1)^{3/2}\frac{dy}{dx} + 3x(x^2+1)^{1/2}y = 6x(x^2+1)^{1/2},$$

and thus

$$D_x\left[(x^2+1)^{3/2}y\right] = 6x(x^2+1)^{1/2}.$$

Integration then yields

$$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx = 2(x^2+1)^{3/2} + C.$$

Multiplication of both sides by $(x^2+1)^{-3/2}$ gives the general solution

$$y(x) = 2 + C(x^2+1)^{-3/2}. \tag{10}$$

∎

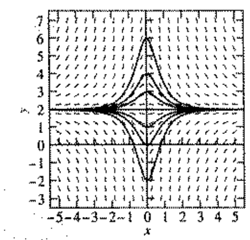

FIGURE 1.5.2. Slope field and solution curves for the differential equation in Eq. (9).

Remark: Figure 1.5.2 shows a slope field and typical solution curves for Eq. (9). Note that, as $x \to +\infty$, all other solution curves approach the constant solution curve $y(x) \equiv 2$ that corresponds to $C = 0$ in Eq. (10). This constant solution can be described as an ***equilibrium solution*** of the differential equation, because $y(0) = 2$ implies that $y(x) = 2$ for all x (and thus the value of the solution remains forever where it starts). More generally, the word "equilibrium" connotes "unchanging," so by an equilibrium solution of a differential equation is meant a constant solution $y(x) \equiv c$, for which it follows that $y'(x) \equiv 0$. Note that substitution of $y' = 0$ in the differential equation (9) yields $3xy = 6x$, so it follows that $y = 2$ if $x \neq 0$. Hence we see that $y(x) \equiv 2$ is the only equilibrium solution of this differential equation, as seems visually obvious in Fig. 1.5.2. ∎

A Closer Look at the Method

The preceding derivation of the solution in Eq. (6) of the linear first-order equation $y' + Py = Q$ bears closer examination. Suppose that the coefficient functions $P(x)$ and $Q(x)$ are continuous on the (possibly unbounded) open interval I. Then the antiderivatives

$$\int P(x)\,dx \quad \text{and} \quad \int \left(Q(x)e^{\int P(x)\,dx}\right)dx$$

exist on I. Our derivation of Eq. (6) shows that *if* $y = y(x)$ is a solution of Eq. (3) on I, *then* $y(x)$ is given by the formula in Eq. (6) for some choice of the constant C. Conversely, you may verify by direct substitution (Problem 31) that the function $y(x)$ given in Eq. (6) satisfies Eq. (3). Finally, given a point x_0 of I and any number y_0, there is—as previously noted—a unique value of C such that $y(x_0) = y_0$. Consequently, we have proved the following existence-uniqueness theorem.

THEOREM 1 The Linear First-Order Equation

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0 \tag{11}$$

has a unique solution $y(x)$ on I, given by the formula in Eq. (6) with an appropriate value of C.

Remark 1: Theorem 1 gives a solution on the *entire* interval I for a *linear* differential equation, in contrast with Theorem 1 of Section 1.3, which guarantees only a solution on a possibly smaller interval.

Remark 2: Theorem 1 tells us that every solution of Eq. (3) is included in the general solution given in Eq. (6). Thus a *linear* first-order differential equation has *no* singular solutions.

Remark 3: The appropriate value of the constant C in Eq. (6)—as needed to solve the initial value problem in Eq. (11)—can be selected "automatically" by writing

$$\rho(x) = \exp\left(\int_{x_0}^{x} P(t)\,dt\right),$$
$$y(x) = \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(t)Q(t)\,dt\right]. \tag{12}$$

The indicated limits x_0 and x effect a choice of indefinite integrals in Eq. (6) that guarantees in advance that $\rho(x_0) = 1$ and that $y(x_0) = y_0$ (as you can verify directly by substituting $x = x_0$ in Eqs. (12)). ■

Example 3 Solve the initial value problem

$$x^2\frac{dy}{dx} + xy = \sin x, \quad y(1) = y_0. \tag{13}$$

Solution Division by x^2 gives the linear first-order equation

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x^2}$$

with $P(x) = 1/x$ and $Q(x) = (\sin x)/x^2$. With $x_0 = 1$ the integrating factor in (12) is

$$\rho(x) = \exp\left(\int_1^x \frac{1}{t}\,dt\right) = \exp(\ln x) = x,$$

so the desired particular solution is given by

$$y(x) = \frac{1}{x}\left(y_0 + \int_1^x \frac{\sin t}{t}\,dt\right). \tag{14}$$

In accord with Theorem 1, this solution is defined on the whole positive x-axis. ■

Comment: In general, an integral such as the one in Eq. (14) would (for given x) need to be approximated numerically—using Simpson's rule, for instance—to find the value $y(x)$ of the solution at x. In this case, however, we have the sine integral function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

which appears with sufficient frequency in applications that its values have been tabulated. A good set of tables of special functions is Abramowitz and Stegun, *Handbook of Mathematical Functions* (New York: Dover, 1965). Then the particular solution in Eq. (14) reduces to

$$y(x) = \frac{1}{x}\left(y_0 + \int_0^x \frac{\sin t}{t}\,dt - \int_0^1 \frac{\sin t}{t}\,dt\right) = \frac{1}{x}\left[y_0 + \mathrm{Si}(x) - \mathrm{Si}(1)\right]. \tag{15}$$

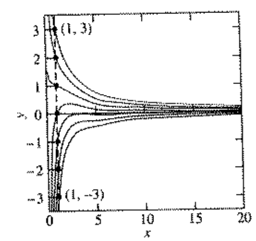

FIGURE 1.5.3. Typical solution curves defined by Eq. (15).

The sine integral function is available in most scientific computing systems and can be used to plot typical solution curves defined by Eq. (15). Figure 1.5.3 shows a selection of solution curves with initial values $y(1) = y_0$ ranging from $y_0 = -3$ to $y_0 = 3$. It appears that on each solution curve, $y(x) \to 0$ as $x \to +\infty$, and this is in fact true because the sine integral function is bounded. ■

In the sequel we will see that it is the exception—rather than the rule—when a solution of a differential equation can be expressed in terms of elementary functions. We will study various devices for obtaining good approximations to the values of the nonelementary functions we encounter. In Chapter 2 we will discuss numerical integration of differential equations in some detail.

Mixture Problems

As a first application of linear first-order equations, we consider a tank containing a solution—a mixture of solute and solvent—such as salt dissolved in water. There is both inflow and outflow, and we want to compute the *amount* $x(t)$ of solute in the tank at time t, given the amount $x(0) = x_0$ at time $t = 0$. Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the solution in the tank—kept thoroughly mixed by stirring—flows out at the constant rate of r_o liters per second.

To set up a differential equation for $x(t)$, we estimate the change Δx in x during the brief time interval $[t, t + \Delta t]$. The amount of solute that flows into the tank during Δt seconds is $r_i c_i\,\Delta t$ grams. To check this, note how the cancellation of dimensions checks our computations:

$$\left(r_i \frac{\text{liters}}{\text{second}}\right)\left(c_i \frac{\text{grams}}{\text{liter}}\right)(\Delta t \text{ seconds})$$

yields a quantity measured in grams.

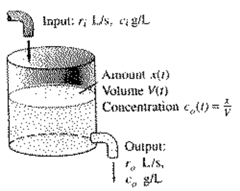

FIGURE 1.5.4. The single-tank mixture problem.

The amount of solute that flows out of the tank during the same time interval depends on the concentration $c_o(t)$ of solute in the solution at time t. But as noted in Fig. 1.5.4, $c_o(t) = x(t)/V(t)$, where $V(t)$ denotes the volume (not constant unless $r_i = r_o$) of solution in the tank at time t. Then

$$\Delta x = \{\text{grams input}\} - \{\text{grams output}\} \approx r_i c_i \,\Delta t - r_o c_o \,\Delta t.$$

We now divide by Δt:

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o.$$

Finally, we take the limit as $\Delta t \to 0$; if all the functions involved are continuous and $x(t)$ is differentiable, then the error in this approximation also approaches zero, and we obtain the differential equation

$$\frac{dx}{dt} = r_i c_i - r_o c_o, \tag{16}$$

in which r_i, c_i, and r_o are constants, but c_o denotes the variable concentration

$$c_o(t) = \frac{x(t)}{V(t)} \tag{17}$$

of solute in the tank at time t. Thus the amount $x(t)$ of solute in the tank satisfies the differential equation

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V}x. \tag{18}$$

If $V_0 = V(0)$, then $V(t) = V_0 + (r_i - r_o)t$, so Eq. (18) is a linear first-order differential equation for the amount $x(t)$ of solute in the tank at time t.

Important: Equation (18) need not be committed to memory. It is the *process* we used to obtain that equation—examination of the behavior of the system over a short time interval $[t, t + \Delta t]$—that you should strive to understand, because it is a very useful tool for obtaining all sorts of differential equations.

Remark: It was convenient for us to use g/L mass/volume units in deriving Eq. (18). But any other consistent system of units can be used to measure amounts of solute and volumes of solution. In the following example we measure both in cubic kilometers. ■

Example 4 Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (years), the pollutant concentration of Lake Erie—caused by past industrial pollution that has now been ordered to cease—is five times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

Solution Here we have

$V = 480$ (km^3),
$r_i = r_o = r = 350$ (km^3/yr),
$c_i = c$ (the pollutant concentration of Lake Huron), and
$x_0 = x(0) = 5cV$,

and the question is this: When is $x(t) = 2cV$? With this notation, Eq. (18) is the separable equation

$$\frac{dx}{dt} = rc - \frac{r}{V}x, \tag{19}$$

which we rewrite in the linear first-order form

$$\frac{dx}{dt} + px = q \tag{20}$$

with constant coefficients $p = r/V$, $q = rc$, and integrating factor $\rho = e^{pt}$. You can either solve this equation directly or apply the formula in (12). The latter gives

$$x(t) = e^{-pt}\left[x_0 + \int_0^t qe^{pt}\,dt\right] = e^{-pt}\left[x_0 + \frac{q}{p}\left(e^{pt} - 1\right)\right]$$

$$= e^{-rt/V}\left[5cV + \frac{rc}{r/V}\left(e^{rt/V} - 1\right)\right];$$

$$x(t) = cV + 4cVe^{-rt/V}. \tag{21}$$

To find when $x(t) = 2cV$, we therefore need only solve the equation

$$cV + 4cVe^{-rt/V} = 2cV \quad \text{for} \quad t = \frac{V}{r}\ln 4 = \frac{480}{350}\ln 4 \approx 1.901 \text{ (years)}.$$ ■

Example 5 A 120-gallon (gal) tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution The interesting feature of this example is that, due to the differing rates of inflow and outflow, the volume of brine in the tank increases steadily with $V(t) = 90 + t$ gallons. The change Δx in the amount x of salt in the tank from time t to time $t + \Delta t$ (minutes) is given by

$$\Delta x \approx (4)(2)\,\Delta t - 3\left(\frac{x}{90+t}\right)\Delta t,$$

so our differential equation is

$$\frac{dx}{dt} + \frac{3}{90+t}x = 8.$$

An integrating factor is

$$\rho(x) = \exp\left(\int \frac{3}{90+t}\,dt\right) = e^{3\ln(90+t)} = (90+t)^3,$$

which gives

$$D_t\left[(90+t)^3 x\right] = 8(90+t)^3;$$
$$(90+t)^3 x = 2(90+t)^4 + C.$$

Substitution of $x(0) = 90$ gives $C = -(90)^4$, so the amount of salt in the tank at time t is

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

The tank is full after 30 min, and when $t = 30$, we have

$$x(30) = 2(90+30) - \frac{90^4}{120^3} \approx 202 \text{ (lb)}$$

of salt in the tank. ■

1.5 Problems

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

1. $y' + y = 2$, $y(0) = 0$
2. $y' - 2y = 3e^{2x}$, $y(0) = 0$
3. $y' + 3y = 2xe^{-3x}$
4. $y' - 2xy = e^{x^2}$
5. $xy' + 2y = 3x$, $y(1) = 5$
6. $xy' + 5y = 7x^2$, $y(2) = 5$
7. $2xy' + y = 10\sqrt{x}$
8. $3xy' + y = 12x$
9. $xy' - y = x$, $y(1) = 7$
10. $2xy' - 3y = 9x^3$
11. $xy' + y = 3xy$, $y(1) = 0$
12. $xy' + 3y = 2x^5$, $y(2) = 1$
13. $y' + y = e^x$, $y(0) = 1$
14. $xy' - 3y = x^3$, $y(1) = 10$
15. $y' + 2xy = x$, $y(0) = -2$
16. $y' = (1 - y)\cos x$, $y(\pi) = 2$
17. $(1 + x)y' + y = \cos x$, $y(0) = 1$
18. $xy' = 2y + x^3\cos x$
19. $y' + y\cot x = \cos x$
20. $y' = 1 + x + y + xy$, $y(0) = 0$
21. $xy' = 3y + x^4\cos x$, $y(2\pi) = 0$
22. $y' = 2xy + 3x^2\exp(x^2)$, $y(0) = 5$
23. $xy' + (2x - 3)y = 4x^4$
24. $(x^2 + 4)y' + 3xy = x$, $y(0) = 1$
25. $(x^2 + 1)\dfrac{dy}{dx} + 3x^3y = 6x\exp\left(-\frac{3}{2}x^2\right)$, $y(0) = 1$

Solve the differential equations in Problems 26 through 28 by regarding y as the independent variable rather than x.

26. $(1 - 4xy^2)\dfrac{dy}{dx} = y^3$
27. $(x + ye^y)\dfrac{dy}{dx} = 1$
28. $(1 + 2xy)\dfrac{dy}{dx} = 1 + y^2$
29. Express the general solution of $dy/dx = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

30. Express the solution of the initial value problem

$$2x\frac{dy}{dx} = y + 2x\cos x, \quad y(1) = 0$$

as an integral as in Example 3 of this section.

Problems 31 and 32 illustrate—for the special case of first-order linear equations—techniques that will be important when we study higher-order linear equations in Chapter 3.

31. **(a)** Show that

$$y_c(x) = Ce^{-\int P(x)\,dx}$$

is a general solution of $dy/dx + P(x)y = 0$. **(b)** Show that

$$y_p(x) = e^{-\int P(x)\,dx}\left[\int\left(Q(x)e^{\int P(x)\,dx}\right)dx\right]$$

is a particular solution of $dy/dx + P(x)y = Q(x)$. **(c)** Suppose that $y_c(x)$ is any general solution of $dy/dx + P(x)y = 0$ and that $y_p(x)$ is any particular solution of $dy/dx + P(x)y = Q(x)$. Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $dy/dx + P(x)y = Q(x)$.

32. **(a)** Find constants A and B such that $y_p(x) = A\sin x + B\cos x$ is a solution of $dy/dx + y = 2\sin x$. **(b)** Use the result of part (a) and the method of Problem 31 to find the general solution of $dy/dx + y = 2\sin x$. **(c)** Solve the initial value problem $dy/dx + y = 2\sin x$, $y(0) = 1$.
33. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture—kept uniform by stirring—is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?
34. Consider a reservoir with a volume of 8 billion cubic feet (ft³) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft³ of water with a pollutant concentration of 0.05% and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?
35. Rework Example 4 for the case of Lake Ontario, which empties into the St. Lawrence River and receives inflow from Lake Erie (via the Niagara River). The only differences are that this lake has a volume of 1640 km³ and an inflow-outflow rate of 410 km³/year.
36. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 h. **(a)** Find the amount of salt in the tank after t minutes. **(b)** What is the maximum amount of salt ever in the tank?
37. A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?
38. Consider the *cascade* of two tanks shown in Fig. 1.5.5, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1. **(a)** Find the amount $x(t)$ of salt in tank 1 at time t. **(b)** Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a). **(c)** Finally, find the maximum amount of salt ever in tank 2.

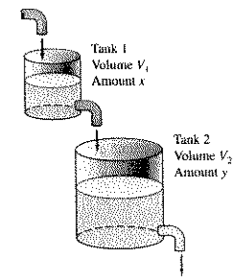

FIGURE 1.5.5. A cascade of two tanks.

39. Suppose that in the cascade shown in Fig. 1.5.5, tank 1 initially contains 100 gal of pure ethanol and tank 2 initially contains 100 gal of pure water. Pure water flows into tank 1 at 10 gal/min, and the other two flow rates are also 10 gal/min. **(a)** Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geqq 0$. **(b)** Find the maximum amount of ethanol ever in tank 2.
40. A multiple cascade is shown in Fig. 1.5.6. At time $t = 0$, tank 0 contains 1 gal of ethanol and 1 gal of water; all the remaining tanks contain 2 gal of pure water each. Pure water is pumped into tank 0 at 1 gal/min, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let $x_n(t)$ denote the amount of ethanol in tank n at time t.

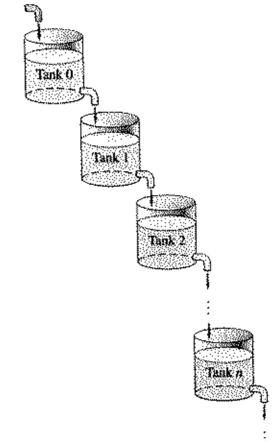

FIGURE 1.5.6. A multiple cascade.

(a) Show that $x_0(t) = e^{-t/2}$. **(b)** Show by induction on n that

$$x_n(t) = \frac{t^n e^{-t/2}}{n!\,2^n} \quad \text{for } n > 0.$$

(c) Show that the maximum value of $x_n(t)$ for $n > 0$ is $M_n = x_n(2n) = n^n e^{-n}/n!$. **(d)** Conclude from **Stirling's approximation** $n! \approx n^n e^{-n}\sqrt{2\pi n}$ that $M_n \approx (2\pi n)^{-1/2}$.

41. A 30-year-old woman accepts an engineering position with a starting salary of \$30,000 per year. Her salary $S(t)$ increases exponentially, with $S(t) = 30e^{t/20}$ thousand dollars after t years. Meanwhile, 12% of her salary is deposited continuously in a retirement account, which accumulates interest at a continuous annual rate of 6%. **(a)** Estimate ΔA in terms of Δt to derive the differential equation satisfied by the amount $A(t)$ in her retirement account after t years. **(b)** Compute $A(40)$, the amount available for her retirement at age 70.
42. Suppose that a falling hailstone with density $\delta = 1$ starts from rest with negligible radius $r = 0$. Thereafter its radius is $r = kt$ (k is a constant) as it grows by accretion during its fall. Use Newton's second law—according to which the net force F acting on a possibly variable mass m equals the time rate of change dp/dt of its momentum $p = mv$—to set up and solve the initial value problem

$$\frac{d}{dt}(mv) = mg, \quad v(0) = 0,$$

where m is the variable mass of the hailstone, $v = dy/dt$ is its velocity, and the positive y-axis points downward. Then show that $dv/dt = g/4$. Thus the hailstone falls as though it were under *one-fourth* the influence of gravity.

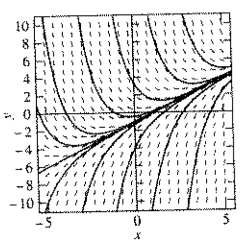

FIGURE 1.5.7. Slope field and solution curves for $y' = x - y$.

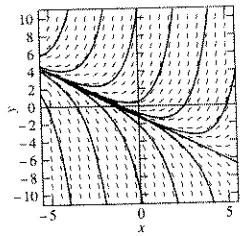

FIGURE 1.5.8. Slope field and solution curves for $y' = x + y$.

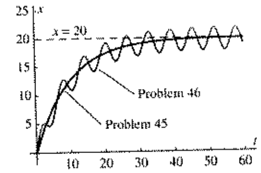

FIGURE 1.5.9. Graphs of solutions in Problems 45 and 46.

43. Figure 1.5.7 shows a slope field and typical solution curves for the equation $y' = x - y$. **(a)** Show that every solution curve approaches the straight line $y = x - 1$ as $x \to +\infty$. **(b)** For each of the five values $y_1 = 3.998$, 3.999, 4.000, 4.001, and 4.002, determine the initial value y_0 (accurate to four decimal places) such that $y(5) = y_1$ for the solution satisfying the initial condition $y(-5) = y_0$.

44. Figure 1.5.8 shows a slope field and typical solution curves for the equation $y' = x + y$. **(a)** Show that every solution curve approaches the straight line $y = -x - 1$ as $x \to -\infty$. **(b)** For each of the five values $y_1 = -10$, -5, 0, 5, and 10, determine the initial value y_0 (accurate to five decimal places) such that $y(5) = y_1$ for the solution satisfying the initial condition $y(-5) = y_0$.

Problems 45 and 46 deal with a shallow reservoir that has a one-square-kilometer water surface and an average water depth of 2 meters. Initially it is filled with fresh water, but at time $t = 0$ water contaminated with a liquid pollutant begins flowing into the reservoir at the rate of 200 thousand cubic meters per month. The well-mixed water in the reservoir flows out at the same rate. Your first task is to find the amount $x(t)$ of pollutant (in millions of liters) in the reservoir after t months.

45. The incoming water has a pollutant concentration of $c(t) = 10$ liters per cubic meter (L/m^3). Verify that the graph of $x(t)$ resembles the steadily rising curve in Fig. 1.5.9, which approaches asymptotically the graph of the equilibrium solution $x(t) \equiv 20$ that corresponds to the reservoir's long-term pollutant content. How long does it take the pollutant concentration in the reservoir to reach 10 L/m^3?

46. The incoming water has pollutant concentration $c(t) = 10(1 + \cos t)$ L/m^3 that varies between 0 and 20, with an average concentration of 10 L/m^3 and a period of oscillation of slightly over $6\frac{1}{4}$ months. Does it seem predictable that the lake's pollutant content should ultimately oscillate periodically about an average level of 20 million liters? Verify that the graph of $x(t)$ does, indeed, resemble the oscillatory curve shown in Fig. 1.5.9. How long does it take the pollutant concentration in the reservoir to reach 10 L/m^3?

1.5 Application Indoor Temperature Oscillations

For an interesting applied problem that involves the solution of a linear differential equation, consider indoor temperature oscillations that are driven by outdoor temperature oscillations of the form

$$A(t) = a_0 + a_1 \cos \omega t + b_1 \sin \omega t. \tag{1}$$

If $\omega = \pi/12$, then these oscillations have a period of 24 hours (so that the cycle of outdoor temperatures repeats itself daily) and Eq. (1) provides a realistic model for the temperature outside a house on a day when no change in the overall day-to-day weather pattern is occurring. For instance, for a typical July day in Athens, Georgia with a minimum temperature of 70°F when $t = 4$ (4 A.M.) and a maximum of 90°F when $t = 16$ (4 P.M.), we would take

$$A(t) = 80 - 10\cos \omega(t - 4) = 80 - 5\cos \omega t - 5\sqrt{3}\sin \omega t. \tag{2}$$

We derived Eq. (2) by using the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to get $a_0 = 80$, $a_1 = -5$, and $b_1 = -5\sqrt{3}$ in Eq. (1).

If we write Newton's law of cooling (Eq. (3) of Section 1.1) for the corresponding indoor temperature $u(t)$ at time t, but with the outside temperature $A(t)$ given by Eq. (1) instead of a constant ambient temperature A, we get the linear first-order differential equation

$$\frac{du}{dt} = -k(u - A(t));$$

that is,

$$\frac{du}{dt} + ku = k(a_0 + a_1 \cos \omega t + b_1 \sin \omega t) \tag{3}$$

with coefficient functions $P(t) \equiv k$ and $Q(t) = kA(t)$. Typical values of the proportionality constant k range from 0.2 to 0.5 (although k might be greater than 0.5 for a poorly insulated building with open windows, or less than 0.2 for a well-insulated building with tightly sealed windows).

SCENARIO: Suppose that our air conditioner fails at time $t_0 = 0$ one midnight, and we cannot afford to have it repaired until payday at the end of the month. We therefore want to investigate the resulting indoor temperatures that we must endure for the next several days.

Begin your investigation by solving Eq. (3) with the initial condition $u(0) = u_0$ (the indoor temperature at the time of the failure of the air conditioner). You may want to use the integral formulas in 49 and 50 of the endpapers, or possibly a computer algebra system. You should get the solution

$$u(t) = a_0 + c_0 e^{-kt} + c_1 \cos \omega t + d_1 \sin \omega t, \tag{4}$$

where

$$c_0 = u_0 - a_0 - \frac{k^2 a_1 - k\omega b_1}{k^2 + \omega^2},$$

$$c_1 = \frac{k^2 a_1 - k\omega b_1}{k^2 + \omega^2}, \quad d_1 = \frac{k\omega a_1 + k^2 b_1}{k^2 + \omega^2}$$

with $\omega = \pi/12$.

With $a_0 = 80$, $a_1 = -5$, $b_1 = -5\sqrt{3}$ (as in Eq. (2)), $\omega = \pi/12$, and $k = 0.2$ (for instance), this solution reduces (approximately) to

$$u(t) = 80 + e^{-t/5}(u_0 - 82.3351) + (2.3351)\cos\frac{\pi t}{12} - (5.6036)\sin\frac{\pi t}{12}. \tag{5}$$

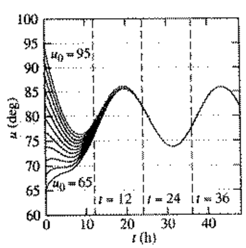

FIGURE 1.5.10. Solution curves given by Eq. (5) with $u_0 = 65, 68, 71, \ldots, 92, 95$.

Observe first that the "damped" exponential term in Eq. (5) approaches zero as $t \to +\infty$, leaving the long-term "steady periodic" solution

$$u_{sp}(t) = 80 + (2.3351)\cos\frac{\pi t}{12} - (5.6036)\sin\frac{\pi t}{12}. \tag{6}$$

Consequently, the long-term indoor temperatures oscillate every 24 hours around the same average temperature 80°F as the average outdoor temperature.

Figure 1.5.10 shows a number of solution curves corresponding to possible initial temperatures u_0 ranging from 65°F to 95°F. Observe that—whatever the initial temperature—the indoor temperature "settles down" within about 18 hours to a periodic daily oscillation. But the amplitude of temperature variation is less indoors than outdoors. Indeed, using the trigonometric identity mentioned earlier, Eq. (6) can be rewritten (verify this!) as

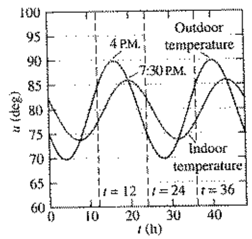

FIGURE 1.5.11. Comparison of indoor and outdoor temperature oscillations.

$$u(t) = 80 - (6.0707)\cos\left(\frac{\pi t}{12} - 1.9656\right)$$

$$= 80 - (6.0707)\cos\frac{\pi}{12}(t - 7.5082). \qquad (7)$$

Do you see that this implies that the indoor temperature varies between a minimum of about 74°F and a maximum of about 86°F?

Finally, comparison of Eqs. (2) and (7) indicates that the indoor temperature lags behind the outdoor temperature by about $7.5082 - 4 \approx 3.5$ hours, as illustrated in Fig. 1.5.11. Thus the temperature inside the house continues to rise until about 7:30 P.M. each evening, so the hottest part of the day inside is early evening rather than late afternoon (as outside).

For a personal problem to investigate, carry out a similar analysis using average July daily maximum/minimum figures for your own locale and a value of k appropriate to your own home. You might also consider a winter day instead of a summer day. (What is the winter-summer difference for the indoor temperature problem?) You may wish to explore the use of available technology both to solve the differential equation and to graph its solution for the indoor temperature in comparison with the outdoor temperature.

1.6 Substitution Methods and Exact Equations

The first-order differential equations we have solved in the previous sections have all been either separable or linear. But many applications involve differential equations that are neither separable nor linear. In this section we illustrate (mainly with examples) substitution methods that sometimes can be used to transform a given differential equation into one that we already know how to solve.

For instance, the differential equation

$$\frac{dy}{dx} = f(x, y), \qquad (1)$$

with dependent variable y and independent variable x, may contain a conspicuous combination

$$v = \alpha(x, y) \qquad (2)$$

of x and y that suggests itself as a new independent variable v. Thus the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2$$

practically demands the substitution $v = x + y + 3$ of the form in Eq. (2).

If the substitution relation in Eq. (2) can be solved for

$$y = \beta(x, v), \qquad (3)$$

then application of the chain rule—regarding v as an (unknown) function of x—yields

$$\frac{dy}{dx} = \frac{\partial\beta}{\partial x}\frac{dx}{dx} + \frac{\partial\beta}{\partial v}\frac{dv}{dx} = \beta_x + \beta_v\frac{dv}{dx}, \qquad (4)$$

where the partial derivatives $\partial\beta/\partial x = \beta_x(x, v)$ and $\partial\beta/\partial v = \beta_v(x, v)$ are *known* functions of x and v. If we substitute the right-hand side in (4) for dy/dx in Eq. (1) and then solve for dv/dx, the result is a *new* differential equation of the form

$$\frac{dv}{dx} = g(x, v) \qquad (5)$$

with *new* dependent variable v. If this new equation is either separable or linear, then we can apply the methods of preceding sections to solve it.

If $v = v(x)$ is a solution of Eq. (5), then $y = \beta(x, v(x))$ will be a solution of the original Eq. (1). The trick is to select a substitution such that the transformed Eq. (5) is one we can solve. Even when possible, this is not always easy; it may require a fair amount of ingenuity or trial and error.

Example 1 Solve the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2.$$

Solution As indicated earlier, let's try the substitution

$$v = x + y + 3; \qquad \text{that is,} \qquad y = v - x - 3.$$

Then

$$\frac{dy}{dx} = \frac{dv}{dx} - 1,$$

so the transformed equation is

$$\frac{dv}{dx} = 1 + v^2.$$

This is a separable equation, and we have no difficulty in obtaining its solution

$$x = \int \frac{dv}{1 + v^2} = \tan^{-1} v + C.$$

So $v = \tan(x - C)$. Because $v = x + y + 3$, the general solution of the original equation $dy/dx = (x + y + 3)^2$ is $x + y + 3 = \tan(x - C)$; that is,

$$y(x) = \tan(x - C) - x - 3.$$

■

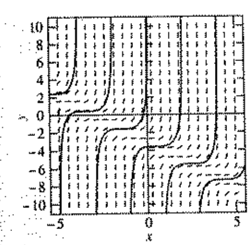

FIGURE 1.6.1. Slope field and solution curves for $y' = (x + y + 3)^2$.

Remark: Figure 1.6.1 shows a slope field and typical solution curves for the differential equation of Example 1. We see that, although the function $f(x, y) = (x+y+3)^2$ is continuously differentiable for all x and y, each solution is continuous only on a bounded interval. In particular, because the tangent function is continuous on the open interval $(-\pi/2, \pi/2)$, the particular solution with arbitrary constant value C is continuous on the interval where $-\pi/2 < x - C < \pi/2$; that is, $C - \pi/2 < x < C + \pi/2$. This situation is fairly typical of nonlinear differential equations, in contrast with linear differential equations, whose solutions are continuous wherever the coefficient functions in the equation are continuous. ■