Forced Damped Vibrations

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Forced Damped Motion

Real systems do not exhibit idealized harmonic motion, because **damping** occurs. A watch balance wheel submerged in oil is a key example: frictional forces due to the viscosity of the oil will cause the wheel to stop after a short time. The same wheel submerged in air will appear to display harmonic motion, but indeed there is friction present, however small, which slows the motion.

Consider a spring–mass system consisting of a mass m and a spring with Hooke's constant k, with an added **dashpot** or **damper**, depicted in Figure 1 as a piston inside a cylinder attached to the mass. A useful physical model, for purposes of intuition, is a screen door with door–closer: the closer has a spring and an adjustable piston–cylinder style damper.

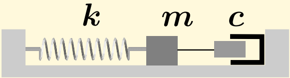

Figure 1. A spring-mass system with damper

Model Derivation

The damper is assumed to operate in the **viscous domain**, which means that the force due to the damper device is proportional to the speed that the mass is moving: $F = cx'(t)$. The number $c \geq 0$ is called the **damping constant**. Three forces act: (1) Newton's second law $F_1 = mx''(t)$, (2) viscous damping $F_2 = cx'(t)$ and (3) the spring restoring force $F_3 = kx(t)$. The sum of the forces $F_1 + F_2 + F_3$ acting on the system must equal the **external force** $f(t)$, which gives the equation for a **damped spring–mass system**

$$mx''(t) + cx'(t) + kx(t) = f(t). \tag{1}$$

Definitions

The motion is called **damped** if $c > 0$ and **undamped** if $c = 0$. If there is no external force, $f(t) = 0$, then the motion is called **free** or **unforced** and otherwise it is called **forced**.

Visualization

A useful visualization for a forced system is a vertical laboratory spring–mass system with damper placed inside a box, which is transported down a washboard road inside an auto trunk. The function $f(t)$ is the vertical oscillation of the auto trunk. The function $x(t)$ is the motion of the mass in response to the washboard road. See Figure 2.

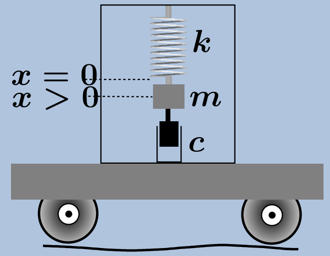

Figure 2. A spring-mass system with damper in a box transported in an auto trunk along a washboard road.

Cafe door

Restaurant workers make trips through a cafe door, which blocks the view of the kitchen – see Figure 3. The door is equipped with a spring which tries to restore the door to the equilibrium position $x = 0$, which is the plane of the door frame. There is a damper attached, to reduce door oscillations.

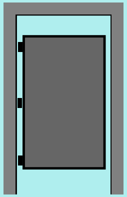

Figure 3. A cafe door on three hinges with damper in the lower hinge. The equilibrium position is the plane of the door frame.

Model Derivation

The top view of the door, Figure 4, shows how the angle $\boldsymbol{x(t)}$ from equilibrium $\boldsymbol{x = 0}$ is measured from different door positions.

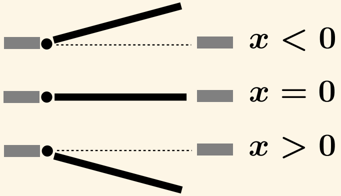

Figure 4. Top view of a cafe door, showing the three possible door positions.

The figure implies that, for modeling purposes, the cafe door can be reduced to a torsional pendulum with viscous damping. This results in the **cafe door** equation

$$\boldsymbol{I x''(t) + c x'(t) + \kappa x(t) = 0.} \tag{2}$$

The removal of the spring ($\boldsymbol{\kappa = 0}$) causes the solution $\boldsymbol{x(t)}$ to be monotonic, which is a reasonable behavior for a springless cafe door.

Pet door

Designed for dogs and cats, the small door in Figure 5 allows animals free entry and exit.

Figure 5. A pet door.
The equilibrium position is the plane of the door frame.

The pet door swings freely from hinges along the top edge. One hinge is spring–loaded with damper. Like the cafe door, the spring restores the door to the equilibrium position while the damper acts to eventually stop the oscillations. However, there is one fundamental difference: if the spring–damper system is removed, then the door continues to oscillate!

The cafe door model will not describe the pet door.

Model Derivation

For modeling purposes, the door can be compressed to a linearized swinging rod of length L (the door height). The torque $I = mL^2/3$ of the door assembly becomes important, as well as the linear restoring force kx of the spring and the viscous damping force cx' of the damper. All considered, a suitable model is the **pet door** equation

$$I\,x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right)x(t) = 0. \tag{3}$$

Derivation of (3) is by equating to zero the algebraic sum of the forces. Removing the damper and spring ($c = k = 0$) gives a harmonic oscillator $x''(t) + \omega^2 x(t) = 0$ with $\omega^2 = 0.5mgL/I$, which establishes sanity for the modeling effort.

Equation (3) is *formally* the cafe door equation with an added linearization term $0.5mgLx(t)$ obtained from $0.5mgL\sin x(t)$.

Damped Free Oscillation Model

All equations can be reduced, for suitable definitions of constants $\boldsymbol{p}$ and $\boldsymbol{q}$, to the simplified second order differential equation

$$\boldsymbol{x''(t) + p\,x'(t) + q\,x(t) = 0.} \tag{4}$$

Tuning a Damper

- The pet door and the cafe door have dampers with an adjustment screw. The screw changes the damping coefficient c which in turn changes the size of coefficient p in (4). More damping c means p is larger.
- There is a *critical damping effect* for a certain screw setting: if the damping is decreased more, then the door *oscillates*, whereas if the damping is increased, then the door has a *monotone non-oscillatory behavior*. The critical effect provides the least time for closing the door. The monotonic behavior can result in the door opening in one direction followed by slowly settling to exactly the door jamb position. If p is too large, then it could take 10 minutes for the door to close!
- The critical case corresponds to the least $p > 0$ (the smallest damping constant $c > 0$) required to close the door with this kind of monotonic behavior. The same can be said about decreasing the damping: the more p is decreased, the more the door oscillations approach those of no damper at all, which is a pure harmonic oscillation.

As viewed from the characteristic equation $r^2 + pr + q = 0$, the change is due to a change in character of the roots from real to complex. The physical response and the three cases in Euler's constant–coefficient recipe lead to the following terminology.

Classification	**Defining properties**
Overdamped	Distinct real roots $r_1 \neq r_2$ Positive discriminant $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $= \text{exponential} \times \text{monotonic function}$
Critically damped	Double real root $r_1 = r_2$ Zero discriminant $x = c_1 e^{r_1 t} + c_2\, t\, e^{r_1 t}$ $= \text{exponential} \times \text{monotonic function}$
Underdamped	Complex conjugate roots $\alpha \pm i\,\beta$ Negative discriminant $x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$ $= \text{exponential} \times \text{harmonic oscillation}$

Distinguishing the Three Damped Models

Consider the pet door or the cafe door, modeled by a differential equation

$$ay'' + by' + cy = 0$$

with constant coefficients a, b, c. We imagine the door closing due to adjustment of the damper screw, which affects the magnitude of coefficient b: small values of b create oscillation and large values non-oscillation.

Classification	Physical Meaning
Overdamped	The door closes slowly without oscillations.
Critically damped	The door closes without oscillations, in the least amount of time.
Underdamped	The door oscillates through the jamb position many times with decreasing amplitudes.

Bicycle trailer

An auto tows a one–wheel trailer over a washboard road. Shown in Figure 6 is the trailer strut, which has a single coil spring and two dampers. The mass m includes the trailer and the bicycles.

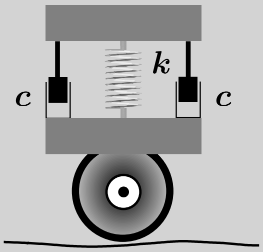

Figure 6. A trailer strut with dampers on a washboard road

Road Surface Model

Suppose a washboard dirt road has about 2 full oscillations (2 bumps and 2 valleys) every 3 meters and a full oscillation has amplitude 6 centimeters. Let s denote the horizontal distance along the road and let ω be the number of full oscillations of the roadway per unit length. The oscillation period is $2\pi/\omega$, therefore $2\pi/\omega = 3/2$ or $\omega = 4\pi/3$.

A model for the road surface is

$$y = \frac{5}{100} \cos \omega s.$$

Model Derivation

Let $x(t)$ denote the vertical elongation of the spring, measured from equilibrium. Newton's second law gives a force $F_1 = mx''(t)$ and the viscous damping force is $F_2 = 2cx'(t)$. The trailer elongates the spring by $x - y$, therefore the Hooke's force is $F_3 = k(x - y)$. The sum of the forces $F_1 + F_2 + F_3$ must be zero, which implies

$$mx''(t) + 2cx'(t) + k(x(t) - y(t)) = 0.$$

Write $s = vt$ where v is the speedometer reading of the car in meters per second. The expanded differential equation is the forced damped spring-mass system equation

$$mx''(t) + 2cx'(t) + kx(t) = \frac{k}{20}\cos(4\pi vt/3).$$

The solution $x(t)$ of this model, with $x(0)$ and $x'(0)$ given, describes the vertical excursion of the trailer bed from the roadway.

The **observed oscillations** of the trailer are modeled by the steady-state solution

$$x_{\text{SS}}(t) = A\cos(4\pi vt/3) + B\sin(4\pi vt/3),$$

where A, B are constants determined by the method of undetermined coefficients. From the physical data, the amplitude $\sqrt{A^2 + B^2}$ of this oscillation might be 6cm or larger.