4.9 Fish Farming

Discussed are logistic models for population dynamics in fish farms. The models are suitable for Pangasius and Tilapia populations. The focus will be on species *tilapia*.

Pangasius. In America, both USA-produced and imported fresh-water catfish can be sold with the labels **Swai**, **Basa** or the subgenus label **Pangasius**, which is the predominant generic label in Europe, with more than 20 varieties. Basa and Swai are different catfish, with different texture and flavor. USA production of farmed catfish increased after 2002, when Vietnam Basa imports were stopped by labeling laws and tariffs. USA channel catfish (four barbels) are harvested after 18 months, at 10 pounds weight. Pangasius varieties are harvested after 4–6 months, at about 2 pounds or less, to produce fillets of 3–12 ounces.
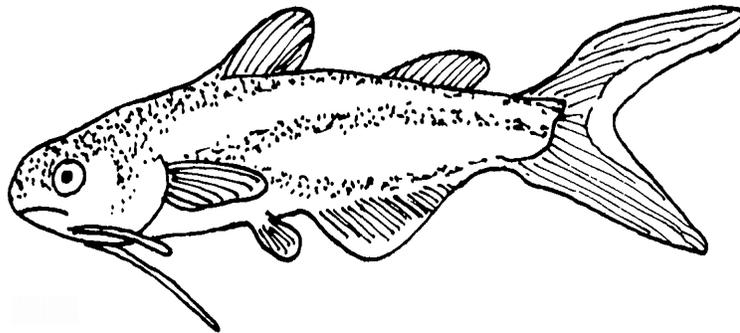

Figure 15. Pangasius, a fresh water catfish with two barbels.

Tilapia. This fresh-water fish originated in Africa 2500 years ago. The popular varieties sold in the USA are marketed under the label **Tilapia** (both dark and light flesh). They are produced in the USA at fish farms in Arizona, California and Florida. Imported Tilapia at 600-900 grams market weight (30% fillets) make up the bulk of USA-consumed Tilapia.
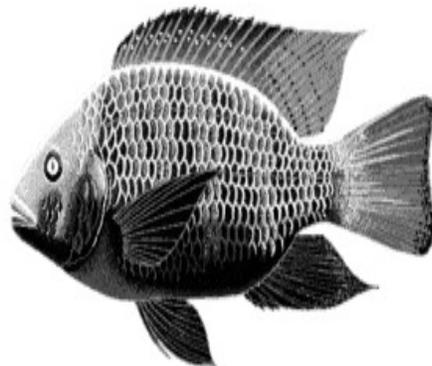

Figure 16. Tilapia.
A fresh water fish from the river Nile. Tilapia are farmed around the world in temperate climates.

Population Dynamics of Fisheries

Fisheries can be wild or farmed. One example is a fish hatchery using concrete tanks. Tilapia freshwater farms can use earthen ponds, canvas tanks, concrete tanks, river cages, pens and old mining quarries.

Tilapia Farming

Detailed life history data for Tilapia is as follows:

- Age at sexual maturity: 5–6 months

- Size at sexual maturity: 28–350 grams

- Stocking ratio for spawning: 7–10 broods/year using 2–5 females per male

- Spawning success: 20–30% spawns per week

- Eggs per female fish: 1–4 eggs per gram of fish

- Survival of egg to fry: 70–90% (fry less than 5 grams)

- Survival of fry to fingerling: 60–90% (fingerling 5–30 grams)

- Survival of fingerling to market: 70–98% (market is 30 to 680 grams)

Tilapia fry might be produced from an initial stock of 1000 female ND-2 and 250 male ND-1. Hatched ND-21 fry will be all male, which have higher market weight. Egg production per female averages from 300 to 500 fry per month, with about 10% lost before reaching 5 gram weight. The marketed Tilapia are about 900 grams in Central America plants (Belize, El Salvador). In Arizona, California and Florida plants, Tilapia market weights vary from 600 to 800 grams, or 1.5–1.75 pounds.

In commercial secondary tanks, fingerlings grow in water temperatures 76–84 degrees Fahrenheit with a death rate of about 0.05%. One fingerling grows to market size on less than 3 pounds of food.

Logistic Harvesting on a Time Interval

The Logistic equation for a **constant harvesting** rate $h \geq 0$ is

$$\frac{dx}{dt} = kx(t)(M - x(t)) - h.$$

The Logistic equation for a non-constant harvesting rate $h(t) \geq 0$ is

$$\frac{dx}{dt} = kx(t)(M - x(t)) - h(t).$$

A simplified situation is constant harvesting $h(t) = c > 0$ on a given time interval $a \leq t \leq b$, but zero otherwise.

In a more sophisticated setting, $h(t)$ is a positive constant c_i on given time interval $a_i \leq t \leq b_i$, $i = 1, \ldots, n$, but zero otherwise. Harvesting can also depend on the population size, which replaces $h(t)$ by $h(t)x(t)$ in the differential equation. Modelling need not be for an individual tank or pond, but the aggregate of all tanks, ponds and cages of an enterprise, viewed from the prospect of so many fish grown to market weight.

Logistic Periodic Harvesting

The periodic harvest Logistic equation is

$$\frac{dx}{dt} = kx(t)(M - x(t)) - h(t)$$

where $h(t) \geq 0$ is the rate of harvest, usually a positive constant c_i on a given time interval $a_i \leq t \leq b_i$, $i = 1, \ldots, n$, but zero otherwise. The equation $h(t + T) = h(t)$ might hold for some value of T, in which case $h(t)$ is a classical periodic function.

Tank harvests can be periodic, in order to reduce the density of fish per volume of water, or to remove fingerlings. Harvested fish can be assumed to be live, and sent either to slaughter or else to another tank, to grow bigger. This model fits Tilapia fry production in ponds, for which it is typical that ND-2 females produce more and more eggs as they mature (then $c_1 < c_2 < c_3 < \cdots$). The time intervals for Tilapia are about a month apart.

Malaysian Tilapia Example

Described here is the 2012 work of M. F. Laham, et al, [**?**], in which a logistic model is used to study harvesting strategies for tilapia fish farming. This work is elementary, in the sense that it treats an ideal example, with no intentional application to management of a Tilapia farm. It illustrates general expectations for fish production, based on gross estimates of a pond scenario.

The data was obtained from the Department of Fisheries of Malaysia and from the Malaysian fish owner of selected ponds situated at Gombak, Selangor. The fisheries department claims (2008) that a fish pond can sustain 5 tilapia fish for every square meter of surface area.[4] The selected pond has an area of 15.61 Hectors, which is equivalent to 156100 square meters, 38 acres or 25000 square feet. The pond carrying capacity is

---

[4]Normal stocking is 1.6 fish per square meter, from which reproduction allows fish population growth to carrying capacity (a theoretical number).

$M = 780500$ fish. Tilapia mature in 6 months and at least 80 percent will survive to maturity (Thomas and Michael 1999 [**?**]).

The Logistic Growth Model, in the absence of harvesting, can be written in the form

$$\frac{dx}{dt} = rx(t)(1 - x(t)/M), \quad r = 0.8, \quad M = 780500.$$

In terms of the alternate model $P' = kP(M - P)$, the constant k equals $rM = 624400$. The work of Laham et al focuses on harvesting strategies, considering the constant harvesting model

$$(1) \qquad\qquad \frac{dx}{dt} = rx(t)(1 - x(t)/M) - H_0$$

and the periodic harvesting model

$$(2) \quad \frac{dy}{dt} = ry(t)(1 - y(t)/M) - H(t), \quad H(t) = \begin{cases} H_0 & 0 \leq t \leq 6, \\ 0 & 6 < t \leq 12. \end{cases}$$

The constant $H_0 = 156100$ is explained below. The discontinuous harvesting function $H(t)$ is extended to be 12-month periodic: $H(t+12) = H(t)$.

Constant Harvesting. The parameters in the model are $r = 0.8$, an estimate of the fraction of fish that will survive to market age, and the pond carrying capacity $M = 780500$. The periodic harvesting value $H_0 = 156100$ arises from the constant harvesting model, by maximizing population size at the equilibrium point for the constant harvesting model. Briefly, the value H_0 is found by requiring $\frac{dx}{dt} = 0$ in the constant harvesting model, replacing $x(t)$ by constant P. This implies

$$(3) \qquad\qquad rP\left(1 - \frac{P}{M}\right) - H_0 = 0.$$

The mysterious value H_0 is the one that makes the discriminant zero in the quadratic formula for P. Then $H_0 = \frac{rM}{4} = 156100$ and $P = 389482$. This **bifurcation point** separates the global behavior of the constant harvesting model as in Table 23. We use the notation P_1, P_2 for the two real equilibrium roots of the quadratic equation (3), assuming $H_0 < 156100$ and $P_1 < P_2$.

Harvest Constant	Initial Population	Behavior
$H_0 = 156100$	$x(0) \geq 389482$	$x(t) \to 389482$,
$H_0 = 156100$	$x(0) < 389482$	$x(t) \to 0$, extinction,
$H_0 > 156100$	any $x(0)$	$x(t) \to 0$, extinction,
$H_0 < 156100$	$x(0) < P_1$	$x(t) \to 0$, extinction,
$H_0 < 156100$	$P_1 < x(0) < P_2$	$x(t) \to P_2$, sustainable,
$H_0 < 156100$	$x(0) \geq P_2$	$x(t) \to P_2$, sustainable.

Table 23. Constant Harvesting Model

Periodic Harvesting. The model is an initial value problem (2) with initial population $y(0)$ equal to the number of Tilapia present, where $t = 0$ is an artificial time representing the current time after some months of growth. The plan is to harvest H_0 fish in the first 6 months.

Direct inspection of the two models shows that $x(t) = y(t)$ for the first six months, regardless of the choice of H_0. Because the constant harvesting model shows that harvesting rates larger than 156100 lead to extinction, then it is clear that the harvesting rate can be $H_0 = 156100$.

The harvesting constant H_0 can be larger than 156100, because the population of fish is allowed to recover for six months after the harvest. Assuming $H_0 > 156100$, then the solution $y(t)$ decreases for 6 months to value $y(6)$, which if positive, allows recovery of the population in the following 6 non-harvest months. There is a catch: the population could fail to grow to harvest size in the following 6 months, causing a reduced production in subsequent years.

To understand the problem more clearly, we present an example where $H_0 > 156100$ and the harvest is sustainable for 3 years, then another example where $H_0 > 156100$ and the harvest fails in the second year.

8 Example (Sustainable Harvest $H_0 > 156100$) Choose $H_0 = 190000$ and $y(0) = 390250 = M/2$. Computer assist gives 6-month population size decreasing to $y(6) = 16028.6$. Then for $6 < t < 12$ the population increases to $y(12) = 560497.2$, enough for a second harvest. The population continues to rise and fall, $y(18) = 320546.6$, $y(24) = 771390.7$, $y(30) = 391554.0$, $y(36) = 774167.6$, a sustainable harvest for the first three years.
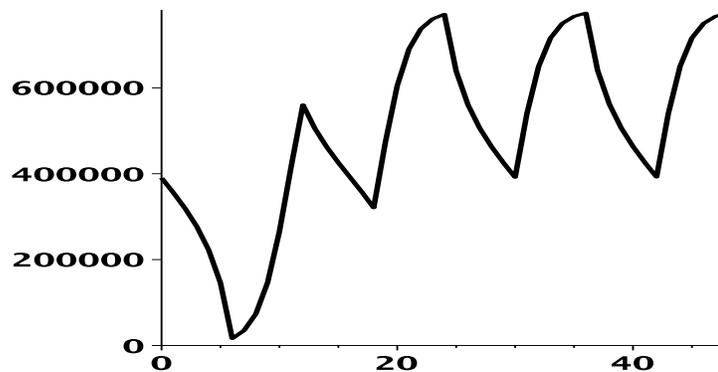

Figure 17. **Sustainable harvest for 3 years, $H_0 = 190000$, $y(0) = M/2$.**

9 Example (Unsustainable Harvest $H_0 > 156100$) Choose $H_0 = 190500$ and $y(0) = 390250 = M/2$. Computer assist gives 6-month population size decreasing to $y(6) = 5263.1$. Then for $6 < t < 12$ the population increases to $y(12) = 352814$, enough for a second harvest. But then the model $y(t)$ decreases to zero (extinction) at $t = 16.95$, meaning the harvest fails in the second year.

The same example with $y(0) = (M/2)(1.02) = 398055$ (2 percent larger) happens to be sustainable for three years. Sustainable harvest is sensitive to both harvesting constant and initial population.
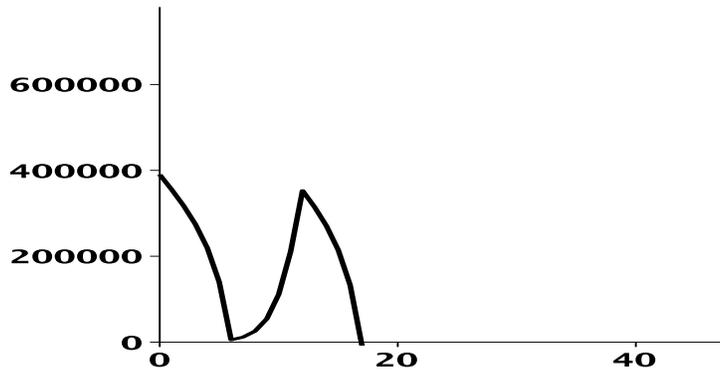

Figure 18. **Unsustainable harvest, failure in year two,** $H_0 = 190500$, $y(0) = M/2$.

Logistic Systems

The Lotka-Volterra equations, also known as the predator-prey equations, are a pair of first order nonlinear differential equations frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey (e.g., foxes and rabbits). They evolve in time according to the pair of equations:

$$\frac{dx}{dt} = x(\alpha - \beta y),$$
$$\frac{dy}{dt} = -y(\gamma - \delta x),$$

where,

x is the number of prey,

y is the number of some predator,

t is time,

$\frac{dy}{dt}$ and $\frac{dx}{dt}$ are population growth rates,

Parameter α is a growth rate for the prey while parameter γ is a death rate for the predator.

Parameters β and δ describe species interaction, with $-\beta xy$ decreasing prey population and δxy increasing predator population.

A. J. Lotka (1910, 1920) used the predator-prey model to study autocatalytic chemical reactions and organic systems such as plants and grazing animals. In 1926, V. Volterra made a statistical analysis of fish catches in the Adriatic Sea, publishing at age 22 the same equations, an independent effort.

Walleye on Lake Erie

The one-dimensional theory of the logistic equation can be applied to fish populations in which there is a predator fish and a prey fish. This problem was studied by A. L. Jensen in 1988. Using the model of P. A. P. Larkin 1966, Jensen invented a mathematical model for walleye populations in the western basin of Lake Erie. The examples for **prey** are Rainbow Smelt (*Osmerus mordax*) in Lake Superior and Yellow Perch (*Perca flavescens*) from Minnesota lakes. The **predator** is Walleye (*Sander vitreus*).

Figure 19.   Yellow Perch.
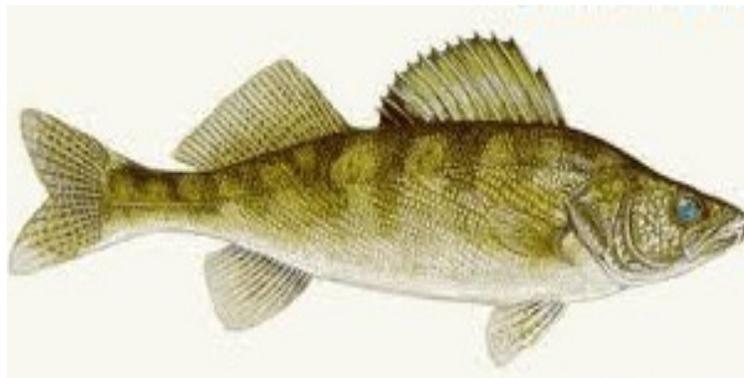The prey, from Shagawa Lake in Northeast Minnesota.

Figure 20.   Walleye.
The predator, also called Yellow Pike, or Pickerel.

The basis for the simulation model is the Lotka-Volterra predator-prey

model. The following assumptions were made.

- A decrease in abundance results in an increase in food concentration.

- An increase in food concentration results in an increase in growth and size.

- An increase in growth and size results in a decrease in mortality because mortality is a function of size.

The relation between prey abundance N_1 and predator abundance N_2 is given by the equations

$$\begin{aligned}
\frac{dN_1}{dt} &= r_l N_1(1 - N_1/K_1) - b_1 N_1 N_2, \\
\frac{dN2}{dt} &= r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2.
\end{aligned}$$

If $b_1 = b_2 = 0$, then there is no interaction of predator and prey, and the two populations N_1, N_2 grow and decay independently of one another. The carrying capacities are K_1, K_2, respectively, because each population N satisfies a logistic equation

$$\frac{dN}{dt} = rN(1 - N/K).$$

Interested readers are referred to the literature below, for further details. Solution methods for systems like (20) are largely numeric. Qualitative methods involving equilibrium points and phase diagrams have an important role in the analysis.

Jensen, A. L.: *Simulation of the potential for life history components to regulate Walleye population size*, Ecological Modelling 45(1), pp 27-41, 1989.

Larkin, P.A.P., 1966. *Exploitation in a type of predator-prey relationship. J. Fish. Res. Board Can.*, 23, pp 349-356, 1966.

Maple Code for Figures 17 and 18

The following sample `maple` code plots the solution on $0 < t < 24$ months with data $H_0 = 190000$, $P_0 = 390250$.

```
de:=diff(P(t),t)=r*(1-P(t)/M)*P(t)-H(t);
r:=0.8:M:=780500:H0:=190000:P0:=M/2:
H:=t->H0*piecewise(t<6,1,t<12,0,t<18,1,0);
DEtools[DEplot](de,P(t),t=0..24,P=0..M,[[P(0)=P0]]);
```

Exercises 4.9

Constant Logistic Harvesting. The model

$$x'(t) = kx(t)(M - x(t)) - h$$

can be converted to the logistic model

$$y'(t) = (a - by(t))y(t)$$

by a change of variables. Find the change of variables $y = x + c$ for the following pairs of equations.

1. $x' = -3x^2 + 8x - 5$,
   $y' = (2 - 3y)y$

2. $x' = -2x^2 + 11x - 14$,
   $y' = (3 - 2y)y$

3. $x' = -5x^2 - 19x - 18$,
   $y' = (1 - 5y)y$

4. $x' = -x^2 + 3x + 4$,
   $y' = (5 - y)y$

Periodic Logistic Harvesting. The periodic harvesting model

$$x'(t) = 0.8x(t)\left(1 - \frac{x(t)}{780500}\right) - H(t)$$

is considered with H defined by

$$H(t) = \begin{cases} 0 & 0 < t < 5, \\ H_0 & 5 < t < 6, \\ 0 & 6 < t < 17, \\ H_0 & 17 < t < 18, \\ 0 & 18 < t < 24. \end{cases}$$

The project is to make a computer graph of the solution on $0 < t < 24$ for various values of H_0 and $x(0)$. See Figures 17 and 18 and the corresponding examples.

5. $H_0 = 156100$, $P(0) = 300000$

6. $H_0 = 156100$, $P(0) = 800000$

7. $H_0 = 800100$, $P(0) = 90000$

8. $H_0 = 800100$, $P(0) = 100000$

von Bertalanffy Equation.

Karl Ludwig von Bertalanffy (1901-1972) derived the equation $\frac{dL}{dt} = r_B(L_\infty - L(t))$ in 1938 from simple physiological arguments. It is a widely used growth curve, especially important in fisheries studies. The symbols:

- t time,
- $L(t)$ length,
- r_B growth rate,
- L_∞ expected length for zero growth.

9. Solve $\frac{dL}{dt} = 2(10 - L)$, $L(0) = 0$. The answer is the length in inches of a fish over time, with final adult size 10 inches.

10. Solve von Bertalanffy's equation to obtain the model

$$L(t) = L_\infty\left(1 - e^{-r_B(t - t_0)}\right).$$

11. Assume von Bertalanffy's model. Suppose field data $L(0) = 0$, $L(1) = 5$, $L(2) = 7$. Display the nonlinear regression details which determine $t_0 = 0$, $L_\infty = 25/3$ and $r_B = \ln(5/2)$.

12. Assume von Bertalanffy's model with field data $L(0) = 0$, $L(1) = 10$, $L(2) = 13$. Find the expected length L_∞ of the fish.