1.2 Exponential Application Library

The model differential equation $y' = ky$, and its variants via a change of variables, appears in various applications to biology, chemistry, finance, science and engineering. All the applications below use the exponential model $y = y_0 e^{kt}$.

Light Intensity	Chemical Reactions
Electric Circuits	Drug Elimination
Drug Dosage	Continuous Interest
Radioactive Decay	Radiocarbon Dating

Light Intensity

Physics defines the **lumen unit** to be the light flux through a solid unit angle from a point source of 1/621 watts of yellow light.[2] The lumen is designed for measuring **brightness**, as perceived by the human eye. The **intensity** $E = \frac{F}{A}$ is the flux F per unit area A, with units Lux or Foot-candles (use $A = 1\mathrm{m}^2$ or $A = 1\mathrm{ft}^2$, respectively). At a radial distance r from a point source, in which case $A = 4\pi r^2$, the intensity is given by the **inverse square law**

$$E = \frac{F}{4\pi r^2}.$$

An **exposure meter**, which measures incident or reflected light intensity, consists of a body, a photocell and a readout in units of Lux or Foot-candles. Light falling on the photocell has energy, which is transferred by the photocell into electrical current and ultimately converted to the readout scale.

In classical physics experiments, a jeweler's bench is illuminated by a source of 8000 lumens. The experiment verifies the inverse square law, by reading an exposure meter at 1/2, 1 and 3/2 meters distance from the source.

As a variant on this experiment, consider a beaker of jeweler's cleaning fluid which is placed over the exposure meter photocell; see Figure 3. Successive meter readings with beaker depths of 0, 5, 10, 15 centimeters show that fluid **absorption** significantly affects the meter readings. Photons[3] striking the fluid convert into heat, which accounts for the rapid loss of intensity at depth in the fluid.

[2]Precisely, the wavelength of the light is 550-nm. The unit is equivalent to one **candela**, one of the seven basic SI units, which is the luminous intensity of one sixtieth of a square centimeter of pure platinum held at 1770C.

[3]A photon is the quantum of electromagnetic radiation, of energy $h\nu$, where ν is the radiation frequency and h is Planck's constant.

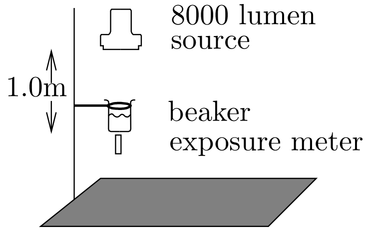


Figure 3. Jeweler's bench experiment.
The exposure meter measures light intensity at the beaker's base.

Empirical evidence from experiments suggests that light intensity $I(x)$ at a depth x in the fluid *changes at a rate proportional to itself*, that is,

$$\frac{dI}{dx} = -kI. \tag{9}$$

If I_0 is the surface intensity at zero depth ($x = 0$) and $I(x)$ is the intensity at depth x meters, then the theory of growth-decay equations applied to equation (9) gives the solution

$$I(x) = I_0 e^{-kx}. \tag{10}$$

Equation (10) says that the intensity $I(x)$ at depth x is a percentage of the surface intensity $I(0) = I_0$, the percentage decreasing with depth x.

Electric Circuits

Classical physics analyzes the RC-circuit in Figure 4 and the LR-circuit in Figure 5. The physics background will be reviewed.

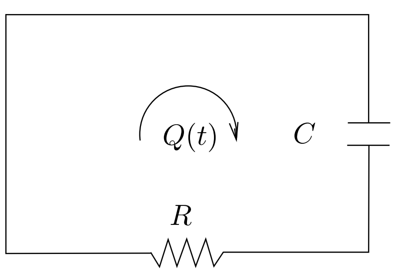


Figure 4. An RC-Circuit, no emf.

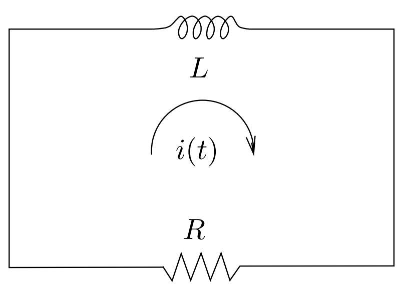


Figure 5. An LR-Circuit, no emf.

First, the **charge** $Q(t)$ in coulombs and the **current** $I(t)$ in amperes are related by the rate formula $I(t) = Q'(t)$. We use prime notation $' = \frac{d}{dt}$. Secondly, there are some empirical laws that are used. There is **Kirchhoff's voltage law**:

> The algebraic sum of the voltage drops around a closed loop is zero.

Kirchhoff's **node law** is not used here, because only one loop appears in the examples.

There are the **voltage drop formulas** for an inductor of L henrys, a resistor of R ohms and a capacitor of C farads:

Faraday's law	$V_L = LI'$
Ohm's law	$V_R = RI$
Coulomb's law	$V_C = Q/C$

In Figure 4, Kirchhoff's law implies $V_R + V_C = 0$. The voltage drop formulas show that the charge $Q(t)$ satisfies $RQ'(t) + (1/C)Q(t) = 0$. Let $Q(0) = Q_0$. Growth-decay theory, page 3, gives $Q(t) = Q_0 e^{-t/(RC)}$.

In Figure 5, Kirchhoff's law implies that $V_L + V_R = 0$. By the voltage drop formulas, $LI'(t) + RI(t) = 0$. Let $I(0) = I_0$. Growth-decay theory gives $I(t) = I_0 e^{-Rt/L}$.

In summary:

RC-**Circuit**	$RQ' + (1/C)Q = 0, \quad Q(0) = Q_0,$
$Q = Q_0 e^{-t/(RC)}$	
LR-**Circuit**	$LI' + RI = 0, \quad I(0) = I_0.$
$I = I_0 e^{-Rt/L}$	

The ideas outlined here are illustrated in Examples 9 and 10, page 22.

Interest

The notion of **simple interest** is based upon the financial formula

$$A = (1 + r)^t A_0$$

where A_0 is the initial amount, A is the final amount, t is the number of years and r is the **annual interest rate** or **rate per annum** (5% means $r = 5/100$). The **compound interest** formula is

$$A = \left(1 + \frac{r}{n}\right)^{nt} A_0$$

where n is the number of times to compound interest per annum. Use $n = 4$ for **quarterly interest** and $n = 360$ for **daily interest**.

The topic of **continuous interest** has its origins in taking the limit as $n \to \infty$ in the compound interest formula. The answer to the limit problem is the **continuous interest formula**

$$A = A_0 e^{rt}$$

which by the growth-decay theory arises from the initial value problem

$$\begin{cases} A'(t) = rA(t), \\ A(0) = A_0. \end{cases}$$

Shown on page 27 are the details for taking the limit as $n \to \infty$ in the compound interest formula. In analogy with population theory, the following statement can be made about continuous interest.

> The amount accumulated by continuous interest increases at a rate proportional to itself.

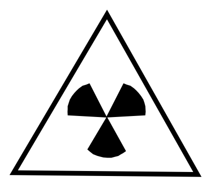
Applied often in interest calculations is the **geometric sum formula**:

$$1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

The reader should pause to verify it for $n = 3$ by expanding the left side of the equivalent identity $(1 + r + r^2 + r^3)(r - 1) = r^4 - 1$.

Radioactive Decay

A constant fraction of the atoms present in a radioactive isotope will spontaneously decay into another isotope of the identical element or else into atoms of another element. Empirical evidence gives the following decay law:



> A radioactive isotope decays at a rate proportional to the amount present.

In analogy with population models the differential equation for radioactive decay is

$$\frac{dA}{dt} = -kA(t),$$

where $k > 0$ is a physical constant called the **decay constant**, $A(t)$ is the number of atoms of radioactive isotope and t is measured in years.

Radiocarbon Dating. The decay constant $k \approx 0.0001245$ is known for carbon-14 (^{14}C). The model applies to measure the date that an organism died, assuming it metabolized atmospheric carbon-14.

The idea of radiocarbon dating is due to Willard S. Libby[4] in the late 1940s. The basis of the chemistry is that radioactive carbon-14, which has two more electrons than stable carbon-12, gives up an electron to become stable nitrogen-14. Replenishment of carbon-14 by cosmic rays keeps atmospheric carbon-14 at a nearly constant ratio with ordinary carbon-12 (this was Libby's assumption). After death, the radioactive decay of carbon-14 depletes the isotope in the organism. The percentage of depletion from atmospheric levels of carbon-14 gives a measurement that dates the organism.

[4] Libby received the Nobel Prize for Chemistry in 1960.

Definition 2 (Half-Life)
The **half-life** of a radioactive isotope is the time T required for half of the isotope to decay. In functional notation, it means $A(T) = A(0)/2$, where $A(t) = A(0)e^{kt}$ is the amount of isotope at time t.

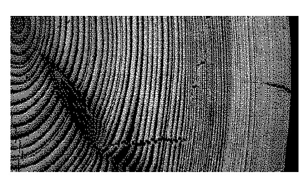
For carbon-14, the half-life is 5568 years plus or minus 30 years, according to Libby (some texts and references give 5730 years). The decay constant $k \approx 0.0001245$ for carbon-14 arises by solving for $k = \ln(2)/5568$ in the equation $A(5568) = \frac{1}{2}A(0)$. Experts believe that carbon-14 dating methods tend to underestimate the age of a fossil.

Uranium-238 undergoes decay via alpha and beta radiation into various nuclides, the half-lives of which are shown in Table 1. The table illustrates the range of possible half-lives for a radioactive substance.

Table 1. Uranium-238 nuclides by alpha or beta radiation.

Nuclide	Half-Life
uranium-238	4,500,000,000 years
thorium-234	24.5 days
protactinium-234	1.14 minutes
uranium-234	233,000 years
thorium-230	83,000 years
radium-236	1,590 years
radon-222	3.825 days
polonium-218	3.05 minutes
lead-214	26.8 minutes
bismuth-214	19.7 minutes
polonium-214	0.00015 seconds
lead-210	22 years
bismuth-210	5 days
polonium-210	140 days
lead-206	stable

Tree Rings. Libby's work was based upon calculations from sequoia tree rings. Later investigations of 4000-year old trees showed that carbon ratios have been non-constant over past centuries.



Libby's method is advertised to be useful for material 200 years to $40,000$ years old. Older material has been dated using the ratio of disintegration byproducts of potassium-40, specifically argon-40 to calcium-40.

An excellent reference for dating methods, plus applications and historical notes on the subject, is Chapter 1 of Braun [**?**].

Chemical Reactions

If the molecules of a substance decompose into smaller molecules, then an empirical law of **first-order reactions** says that the decomposition rate is proportional to the amount of substance present. In mathematical notation, this means

$$\frac{dA}{dt} = -hA(t)$$

where $A(t)$ is the amount of the substance present at time t and h is a physical constant called the **reaction constant**.

The **law of mass action** is used in chemical kinetics to describe **second-order reactions**. The law describes the amount $X(t)$ of chemical C produced by the combination of two chemicals A and B. A chemical derivation produces a rate equation

$$X' = k(\alpha - X)(\beta - X), \quad X(0) = X_0, \tag{11}$$

where k, α and β are physical constants, $\alpha < \beta$; see Zill-Cullen [Z-C], Chapter 2. The substitution $u = (\alpha - X)/(\beta - X)$ is known to transform (11) into $u' = k(\alpha - \beta)u$ (see page 11 for the technique and the exercises in this section). Therefore,

$$X(t) = \frac{\alpha - \beta u(t)}{1 - u(t)}, \quad u(t) = u_0 e^{(\alpha-\beta)kt}, \quad u_0 = \frac{\alpha - X_0}{\beta - X_0}. \tag{12}$$

Drug Elimination

Some drugs are eliminated from the bloodstream by an animal's body in a predictable fashion. The amount $D(t)$ in the bloodstream declines at a rate proportional to the amount already present. Modeling drug elimination exactly parallels radioactive decay, in that the translated mathematical model is

$$\frac{dD}{dt} = -hD(t),$$

where $h > 0$ is a physical constant, called the **elimination constant** of the drug.

Oral drugs must move through the digestive system and into the gut before reaching the bloodstream. The model $D'(t) = -hD(t)$ applies only after the drug has reached a stable concentration in the bloodstream and the body begins to eliminate the drug.

Examples

8 Example (Light intensity) Light intensity in a lake is decreased by 75% at depth one meter. At what depth is the intensity decreased by 95%?

Solution: The answer is 2.16 meters (7 feet, $1\frac{1}{16}$ inches). This depth will be justified by applying the light intensity model $I(x) = I_0 e^{-kx}$, where I_0 is the surface light intensity.

At one meter the intensity is $I(1) = I_0 e^{-k}$, but also it is given as $0.25 I_0$. The equation $e^{-k} = 0.25$ results, to determine $k = \ln 4 \approx 1.3862944$. To find the depth x when the intensity has decreased by 95%, solve $I(x) = 0.05 I_0$ for x. The value I_0 cancels from this equation, leaving $e^{-kx} = 1/20$. The usual logarithm methods give $x \approx 2.2$ meters, as follows:

$\ln e^{-kx} = \ln(1/20)$	Take the logarithm across $e^{-kx} = 1/20$.
$-kx = -\ln(20)$	Use $\ln e^u = u$ and $-\ln u = \ln(1/u)$.
$x = \dfrac{\ln(20)}{k}$	Divide by $-k$.
$= \dfrac{\ln(20)}{\ln(4)}$	Use $k = \ln(4)$.
≈ 2.16 meters.	Only 5% of the surface intensity remains at 2.16 meters.

9 Example (RC-Circuit) Solve the RC-circuit equation $RQ' + (1/C)Q = 0$ when $R = 2$, $C = 10^{-2}$ and the voltage drop across the capacitor at $t = 0$ is 1.5 volts.

Solution: The charge is $Q = 0.015e^{-50t}$.

To justify this equation, start with the voltage drop formula $V_C = Q/C$, page 18. Then $1.5 = Q(0)/C$ implies $Q(0) = 0.015$. The differential equation is $Q' + 50Q = 0$. The solution from page 3 is $Q = Q(0)e^{-50t}$. Then the equation for the charge in coulombs is $Q(t) = 0.015e^{-50t}$.

10 Example (LR-Circuit) Solve the LR-circuit equation $LI' + RI = 0$ when $R = 2$, $L = 0.1$ and the resistor voltage drop at $t = 0$ is 1.0 volts.

Solution: The solution is $I = 0.5e^{-20t}$. To justify this equation, start with the voltage drop formula $V_R = RI$, page 18. Then $1.0 = RI(0)$ implies $I(0) = 0.5$. The differential equation is $I' + 20I = 0$; page 3 gives the solution $I = I(0)e^{-20t}$.

11 Example (Compound Interest) Compute the fixed monthly payment for a 5-year auto loan of $18,000 at 9% per annum, using (a) daily interest and (b) continuous interest.

Solution: The payments are (a) $373.94 and (b) $373.95, which differ by one cent; details below.

Let $A_0 = 18000$ be the initial amount. It will be assumed that the first payment is due after 30 days and monthly thereafter. To simplify the calculation, a **day** is defined to be 1/360th of a year, regardless of the number of days in that year, and payments are applied every 30 days. Late fees apply if the payment is not received within the **grace period**, but it will be assumed here that all payments are made on time.

Part (a). The **daily interest rate** is $R = 0.09/360$ applied for 1800 periods (5 years). Between payments P, daily interest is applied to the balance $A(t)$ owed after t periods. The balance grows between payments and then decreases on the day of the payment. The problem is to *find P so that* $A(1800) = 0$.

Payments are subtracted every 30 periods making balance $A(30k)$. Let $B = (1+R)^{30}$ and $A_k = A(30k)$. Then

$A_k = A(30k)$	Balance after k months.
$\quad = A_0B^k - P(1 + \cdots + B^{k-1})$	For $k = 1, 2, 3, \ldots$
$\quad = A_0B^k - P\dfrac{B^k - 1}{B - 1}$	Geometric sum formula applied, page 19.
$A_0B^{60} = P\dfrac{B^{60} - 1}{B - 1}$	Use $A(1800) = 0$, which corresponds to $k = 60$.
$P = A_0(B - 1)\dfrac{B^{60}}{B^{60} - 1}$	Solve for P.
$\quad = 373.93857$	By calculator.

Part (b). The details are the same except for the method of applying interest. Let $s = 30(0.09)/360$, then

$A_k = A_0e^{ks} - Pe^{ks-s}\left(1 + e^{-s} + \cdots + e^{-ks+s}\right)$	For $k = 1, 2, 3, \ldots$, by examination of cases $A(30)$ and $A(60)$.
$\quad = A_0e^{ks} - Pe^{ks-s}\left(\dfrac{e^{-ks} - 1}{e^{-s} - 1}\right)$	Apply the geometric sum formula with common ratio e^{-s}.
$A_0e^{60s} = Pe^{60s-s}\dfrac{e^{-60s} - 1}{e^{-s} - 1}$	Set $k = 60$ and $A(1800) = 0$ in the previous formula.
$P = A_0\dfrac{-e^s + 1}{e^{-60s} - 1}$	Solve for P.
$\quad = 373.94604$	By calculator.

12 Example (Effective Annual Yield) A bank advertises an effective annual yield of 5.73% for a certificate of deposit with continuous interest rate 5.5% per annum. Justify the rate.

Solution: The **effective annual yield** is the simple annual interest rate which gives the same account balance after one year. The issue is whether one year means 365 days or 360 days, since banks do business on a 360-day cycle.

Suppose first that one year means 365 days. The model used for a saving account is $A(t) = A_0e^{rt}$ where $r = 0.055$ is the interest rate per annum. For one year, $A(1) = A_0e^r$. Then $e^r = 1.0565406$, that is, the account has increased in one year by 5.65%. The *effective annual yield* is 0.0565 or 5.65%.

Suppose next that one year means 360 days. Then the bank pays 5.65% for only 360 days to produce a balance of $A_1 = A_0e^r$. The extra 5 days make 5/360 years, therefore the bank records a balance of $A_1e^{5r/360}$ which is $A_0e^{365r/360}$. The rate for 365 days is then 5.73%, by the calculation

$$\frac{365}{360}0.0565406 = 0.057325886.$$

13 Example (Retirement Funds) An engineering firm offers a starting salary of 40 thousand per year, which is expected to increase 3% per year. Retirement contributions are 11% of salary, deposited monthly, growing at 6% continuous interest per annum. The company advertises a million dollars in retirement funds after 40 years. Justify the claim.

Solution: The salary is estimated to be $S(t) = 40000(1.03)^t$ after t years, because it starts with $S(0) = 40000$ and each year it takes a 3% increment. After 39 years of increases the salary has increased from \$40,000 to \$126,681.

Let A_n be the amount in the retirement account at the end of year n. Let $P_n = (40000(1.03)^n)(0.11)/12$ be the monthly salary for year $n+1$. The interest rates are $r = 0.06$ (annual) and $s = 0.06/12$ (monthly). For brevity, let $R = 1.03$.

During the first year, the retirement account accumulates 12 times for a total

$A_1 = P_0 + \cdots + P_0 e^{11s}$	Continuous interest at rate s on amount P_0 for 1 through 11 months.
$= P_0 \dfrac{e^r - 1}{e^s - 1}$	Geometric sum with common ratio e^s.
$= 4523.3529.$	Retirement balance after one year.

During the second and later years the retirement account accumulates by the rule

$A_{n+1} = A_n e^r + P_n + P_n e^s + \cdots + P_n e^{11s}$	One year's accumulation at continuous rate r on amount A_n plus monthly accumulations on retirement contributions P_n.
$= A_n e^r + P_n \dfrac{e^r - 1}{e^s - 1}$	Apply the geometric sum formula with common ratio e^s.
$= A_n e^r + R^n P_0 \dfrac{e^r - 1}{e^s - 1}$	Use $P_n = P_0 R^n$.
$= A_n e^r + R^n A_1.$	Apply $A_1 = P_0 \dfrac{e^r - 1}{e^s - 1}$.

After examining cases $n = 1, 2, 3$, the recursion is solved to give

$$A_n = A_1 \sum_{k=0}^{n-1} e^{kr} R^{n-1-k}.$$

To establish this formula, induction is applied:

$A_{n+1} = A_n e^r + R^n A_1$	Derived above.
$= A_1 e^r \sum_{k=0}^{n-1} e^{kr} R^{n-1-k} + R^n A_1$	Apply the induction hypothesis.
$= A_1 \sum_{k=0}^{n} e^{kr} R^{n-k}$	Rewrite the sum indices.
$= A_1 R^n \dfrac{(e^r/R)^{n+1} - 1}{e^r/R - 1}.$	Use the geometric sum formula with common ratio e^r/R.

The advertised retirement fund after 40 years should be the amount A_{40}, which is obtained by setting $n = 39$ in the last equality. Then $A_{40} = 1102706.60$.

14 Example (Half-life of Radium) A radium sample loses $1/2$ percent due to disintegration in 12 years. Verify the half-life of the sample is about $1,660$ years.

Solution: The decay model $A(t) = A_0e^{-kt}$ applies. The given information $A(12) = 0.995A(0)$ reduces to the exponential equation $e^{-12k} = 0.995$ with solution $k = \ln(1000/995)/12$. The half-life T satisfies $A(T) = \frac{1}{2}A(0)$, which reduces to $e^{-kT} = 1/2$. Since k is known, the value T can be found as $T = \ln(2)/k \approx 1659.3909$ years.

15 Example (Radium Disintegration) The disintegration reaction

$$_{88}R^{226} \longrightarrow {}_{88}R^{224}$$

of radium-226 into radon has a half-life of 1700 years. Compute the decay constant k in the decay model $A' = -kA$.

Solution: The half-life equation is $A(1700) = \frac{1}{2}A(0)$. Since $A(t) = A_0e^{-kt}$, the equation reduces to $e^{-1700k} = 1/2$. The latter is solved for k by logarithm methods (see page 8), giving $k = \ln(2)/1700 = 0.00040773364$.

16 Example (Radiocarbon Dating) The ratio of carbon-14 to carbon-12 in a dinosaur fossil is 6.34 percent of the current atmospheric ratio. Verify the dinosaur's death was about $22,160$ years ago.

Solution: The method due to Willard Libby will be applied, using his assumption that the ratio of carbon-14 to carbon-12 in living animals is equal to the atmospheric ratio. Then carbon-14 depletion in the fossil satisfies the decay law $A(t) = A_0e^{-kt}$ for some parameter values k and A_0.

Assume the half-life of carbon-14 is 5568 years. Then $A(5568) = \frac{1}{2}A(0)$ (see page 20). This equation reduces to $A_0e^{-5568k} = \frac{1}{2}A_0e^0$ or $k = \ln(2)/5568$. In short, k is known but A_0 is unknown. It is not necessary to determine A_0 in order to do the verification.

At the time t_0 in the past when the organism died, the amount A_1 of carbon-14 began to decay, reaching the value $6.34A_1/100$ at time $t = 0$ (the present). Therefore, $A_0 = 0.0634A_1$ and $A(t_0) = A_1$. Taking this last equation as the starting point, the final calculation proceeds as follows.

$A_1 = A(t_0)$	The amount of carbon-14 at death is A_1, $-t_0$ years ago.
$= A_0e^{-kt_0}$	Apply the decay model $A = A_0e^{-kt}$ at $t = t_0$.
$= 0.0634A_1e^{-kt_0}$	Use $A_0 = 6.34A_1/100$.

The value A_1 cancels to give the new relation $1 = 0.0634e^{-kt_0}$. The value $k = \ln(2)/5568$ gives an exponential equation to solve for t_0:

$e^{kt_0} = 0.0634$	Multiply by e^{kt_0} to isolate the exponential.
$\ln e^{kt_0} = \ln(0.0634)$	Take the logarithm of both sides.
$t_0 = \frac{1}{k}\ln(0.0634)$	Apply $\ln e^u = u$ and divide by k.
$= \frac{5568}{\ln 2}\ln(0.0634)$	Substitute $k = \ln(2)/5568$.
$= -22157.151$ years.	By calculator. The fossil's age is the negative.

17 **Example (Percentage of an Isotope)** A radioactive isotope disintegrates by 5% in ten years. By what percentage does it disintegrate in one hundred years?

Solution: The answer is not 50%, as is widely reported by lay persons. The correct answer is 40.13%. It remains to justify this non-intuitive answer.

The model for decay is $A(t) = A_0 e^{-kt}$. The decay constant k is known because of the information *. . . disintegrates by 5% in ten years.* Translation to equations produces $A(10) = 0.95A_0$, which reduces to $e^{-10k} = 0.95$. Solving with logarithms gives $k = 0.1\ln(100/95) \approx 0.0051293294$.

After one hundred years, the isotope present is $A(100)$, and the percentage is $100\frac{A(100)}{A(0)}$. The common factor A_0 cancels to give the percentage $100e^{-100k} \approx 59.87$. The reduction is 40.13%.

To reconcile the lay person's answer, observe that the amounts present after one, two and three years are $0.95A_0$, $(0.95)^2 A_0$, $(0.95)^3 A_0$. The lay person should have guessed 100 times $1 - (0.95)^{10}$, which is 40.126306. The common error is to simply multiply 5% by the ten periods of ten years each. By this erroneous reasoning, the isotope would be depleted in two hundred years, whereas the decay model says that about 36% of the isotope remains!

18 **Example (Chemical Reaction)** The manufacture of t-butyl alcohol from t-butyl chloride is made by the chemical reaction

$$(CH_3)_3CCL + NaOH \longrightarrow (CH_3)_3COH + NaCL.$$

Model the production of t-butyl alcohol, when N% of the chloride remains after t_0 minutes.

Solution: It will be justified that the model for alcohol production is $A(t) = C_0(1 - e^{-kt})$ where $k = \ln(100/N)/t_0$, C_0 is the initial amount of chloride and t is in minutes.

According to the theory of first-order reactions, the model for chloride depletion is $C(t) = C_0 e^{-kt}$ where C_0 is the initial amount of chloride and k is the reaction constant. The alcohol production is $A(t) = C_0 - C(t)$ or $A(t) = C_0(1 - e^{-kt})$. The reaction constant k is found from the initial data $C(t_0) = \frac{N}{100}C_0$, which results in the exponential equation $e^{-kt_0} = N/100$. Solving the exponential equation gives $k = \ln(100/N)/t_0$.

19 Example (Drug Dosage) A veterinarian applies general anesthesia to animals by injection of a drug into the bloodstream. Predict the drug dosage to anesthetize a 25-pound animal for thirty minutes, given:

1. The drug requires an injection of 20 milligrams per pound of body weight in order to work.
2. The drug eliminates from the bloodstream at a rate proportional to the amount present, with a half-life of 5 hours.

Solution: The answer is about 536 milligrams of the drug. This amount will be justified using exponential modeling.

The drug model is $D(t) = D_0 e^{-ht}$, where D_0 is the initial dosage and h is the elimination constant. The half-life information $D(5) = \frac{1}{2}D_0$ determines $h = \ln(2)/5$. Depletion of the drug in the bloodstream means the drug levels are always decreasing, so it is enough to require that the level at 30 minutes exceeds 20 times the body weight in pounds, that is, $D(1/2) > (20)(25)$. The critical value of the initial dosage D_0 then occurs when $D(1/2) = 500$ or $D_0 = 500e^{h/2} = 500e^{0.1\ln(2)}$, which by calculator is approximately 535.88673 milligrams.

Drugs like sodium pentobarbital behave somewhat like this example, although injection in a single dose is unusual. An intravenous drip can sustain the blood levels of the drug, keeping the level closer to the target 500 milligrams.

Details and Proofs

Verification of Continuous Interest by Limiting. Derived here is the continuous interest formula by limiting as $n \to \infty$ in the compound interest formula.

$\left(1+\frac{r}{n}\right)^{nt} = B^{nt}$	In the exponential rule $B^x = e^{x\ln B}$, the base is $B = 1 + r/n$.
$= e^{nt\ln B}$	Use $B^x = e^{x\ln B}$ with $x = nt$.
$= e^{\frac{r\ln(1+u)}{u}\,t}$	Substitute $u = r/n$. Then $u \to 0$ as $n \to \infty$.
$\approx e^{rt}$	Because $\ln(1+u)/u \approx 1$ as $u \to 0$, by L'Hospital's rule.

Exercises 1.2

Light Intensity. The following exercises apply the theory of light intensity on page 16, using the model $I(t) = I_0 e^{-kx}$ with x in meters. Methods parallel Example 8 on page 21.

1. The light intensity is $I(t) = I_0 e^{-1.4x}$ in a certain swimming pool. At what depth does the light intensity fall off by 50%?

2. The light intensity in a swimming pool falls off by 50% at a depth of 2.5 meters. Find the depletion constant k in the exponential model.

3. Plastic film is used to cover window glass, which reduces the interior light intensity by 10%. By what percentage is the intensity reduced, if two layers are used?

4. Double-thickness colored window glass is supposed to reduce the interior light intensity by 20%. What is the reduction for single-thickness colored glass?

RC-Electric Circuits. In the exercises below, solve for $Q(t)$ when $Q_0 = 10$ and graph $Q(t)$ on $0 \le t \le 5$.

5. $R = 1$, $C = 0.01$.

6. $R = 0.05$, $C = 0.001$.

7. $R = 0.05$, $C = 0.01$.

8. $R = 5$, $C = 0.1$.

9. $R = 2$, $C = 0.01$.

10. $R = 4$, $C = 0.15$.

11. $R = 4$, $C = 0.02$.

12. $R = 50$, $C = 0.001$.

LR-Electric Circuits. In the exercises below, solve for $I(t)$ when $I_0 = 5$ and graph $I(t)$ on $0 \le t \le 5$.

13. $L = 1$, $R = 0.5$.

14. $L = 0.1$, $R = 0.5$.

15. $L = 0.1$, $R = 0.05$.

16. $L = 0.01$, $R = 0.05$.

17. $L = 0.2$, $R = 0.01$.

18. $L = 0.03$, $R = 0.01$.

19. $L = 0.05$, $R = 0.005$.

20. $L = 0.04$, $R = 0.005$.

Interest and Continuous Interest. Financial formulas which appear on page 18 are applied below, following the ideas in Examples 11, 12 and 13, pages 22–24.

21. **(Total Interest)** Compute the total daily interest and also the total continuous interest for a 10-year loan of 5,000 dollars at 5% per annum.

22. **(Total Interest)** Compute the total daily interest and also the total continuous interest for a 15-year loan of 7,000 dollars at $5\frac{1}{4}$% per annum.

23. **(Monthly Payment)** Find the monthly payment for a 3-year loan of 8,000 dollars at 7% per annum compounded continuously.

24. **(Monthly Payment)** Find the monthly payment for a 4-year loan of 7,000 dollars at $6\frac{1}{3}$% per annum compounded continuously.

25. **(Effective Yield)** Determine the effective annual yield for a certificate of deposit at $7\frac{1}{4}$% interest per annum, compounded continuously.

26. **(Effective Yield)** Determine the effective annual yield for a certificate of deposit at $5\frac{3}{4}$% interest per annum, compounded continuously.

27. **(Retirement Funds)** Assume a starting salary of 35,000 dollars per year, which is expected to increase 3% per year. Retirement contributions are $10\frac{1}{2}$% of salary, deposited monthly, growing at $5\frac{1}{2}$% continuous interest per annum. Find the retirement amount after 30 years.

28. **(Retirement Funds)** Assume a starting salary of $45,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $9\frac{1}{2}\%$ of salary, deposited monthly, growing at $6\frac{1}{4}\%$ continuous interest per annum. Find the retirement amount after 30 years.

29. **(Actual Cost)** A van is purchased for $18,000$ dollars with no money down. Monthly payments are spread over 8 years at $12\frac{1}{2}\%$ interest per annum, compounded continuously. What is the actual cost of the van?

30. **(Actual Cost)** Furniture is purchased for $15,000$ dollars with no money down. Monthly payments are spread over 5 years at $11\frac{1}{8}\%$ interest per annum, compounded continuously. What is the actual cost of the furniture?

Radioactive Decay. Assume the decay model $A' = -kA$ from page 19. Below, $A(T) = 0.5A(0)$ defines the *half-life* T. Methods parallel Examples 14– 17 on pages 25– 26.

31. **31.**(Half-Life) Determine the half-life of a radium sample which decays by 5.5% in 13 years.

32. **(Half-Life)** Determine the half-life of a radium sample which decays by 4.5% in 10 years.

33. **(Half-Life)** Assume a radioactive isotope has half-life 1800 years. Determine the percentage decayed after 150 years.

34. **(Half-Life)** Assume a radioactive isotope has half-life 1650 years. Determine the percentage decayed after 99 years.

35. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 5.5% in 13 years.

36. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 4.5% in 10 years.

37. **(Radiocarbon Dating)** A fossil found near the town of Dinosaur, Utah contains carbon-14 at a ratio of 6.21% to the atmospheric value. Determine its approximate age according to Libby's method.

38. **(Radiocarbon Dating)** A fossil found in Colorado contains carbon-14 at a ratio of 5.73% to the atmospheric value. Determine its approximate age according to Libby's method.

39. **(Radiocarbon Dating)** In 1950, the Lascaux Cave in France contained charcoal with 14.52% of the carbon-14 present in living wood samples nearby. Estimate by Libby's method the age of the charcoal sample.

40. **(Radiocarbon Dating)** At an excavation in 1960, charcoal from building material had 61% of the carbon-14 present in living wood nearby. Estimate the age of the building.

41. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 5% in 12 years. By what percentage is it reduced in 99 years?

42. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 6.5% in $1,000$ years. By what percentage is it reduced in $5,000$ years?

Chemical Reactions. Assume below the model $A' = kA$ for a first-order reaction. See page 21 and Example 18, page 26.

43. **(First-Order $A+B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $N\%$ of A remains after t_0 minutes.

44. **(First-Order $A+B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $M\%$ of A is depleted after t_0 minutes.

45. **(Law of Mass-Action)** Consider a second-order chemical reaction $X(t)$ with $k = 0.14$, $\alpha = 1$, $\beta = 1.75$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 2$.

46. **(Law of Mass-Action)** Consider a second-order chemical reaction $X(t)$ with $k = 0.015$, $\alpha = 1$, $\beta = 1.35$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 10$.

47. **(Mass-Action Derivation)** Let k, α, β be positive constants, $\alpha < \beta$. Solve $X' = k(\alpha - X)(\beta - X)$, $X(0) = X_0$ by the substitution $u = (\alpha - X)/(\beta - X)$, showing that $X = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha-\beta)kt}$, $u_0 = (\alpha - X_0)/(\beta - X_0)$.

48. **(Mass-Action Derivation)** Let k, α, β be positive constants, $\alpha < \beta$. Define $X = (\alpha - \beta u)/(1 - u)$, where $u = u_0 e^{(\alpha-\beta)kt}$ and $u_0 = (\alpha - X_0)/(\beta - X_0)$. Verify by calculus computation that (1) $X' = k(\alpha - X)(\beta - X)$ and (2) $X(0) = X_0$.

Drug Dosage. Employ the drug dosage model $D(t) = D_0 e^{-ht}$ given on page 21. Let h be determined by a half-life of three hours. Apply the techniques of Example 19, page 27.

49. **(Injection Dosage)** Bloodstream injection of a drug into an animal requires a minimum of 20 milligrams per pound of body weight. Predict the dosage for a 12-pound animal which will maintain a drug level 3% higher than the minimum for two hours.

50. **(Injection Dosage)** Bloodstream injection of an antihistamine into an animal requires a minimum of 4 milligrams per pound of body weight. Predict the dosage for a 40-pound animal which will maintain an antihistamine level 5% higher than the minimum for twelve hours.

51. **(Oral Dosage)** An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Predict the number of hours after the first dose at which to take a second dose, in order to maintain a blood level of at least 180 milligrams for three hours.

52. **(Oral Dosage)** An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Determine three (small) dosage amounts, and their administration time, which keep the blood level above 180 milligrams but below 280 milligrams over three hours.