Differential Equations 2280
Final Exam
Thursday, 28 April 2016, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

(a) [60%] Solve $2v'(t) = 5 + \frac{1}{t+1}v(t)$, $v(0) = 5$. Show all integrating factor steps.

(b) [20%] Solve the linear homogeneous equation $2\sqrt{x+1}\,\frac{dy}{dx} = 2y$.

(c) [20%] The problem $2\sqrt{x+1}\,y' = 2y - 5$ can be solved using superposition $y = y_h + y_p$. Find y_h and y_p.

Answer:
(a) The equilibrium solutions are $y = 1$, $y = -1$. The implicit solution is

$$-\frac{1}{2(y^2-1)} = x^2 - x + \ln|1+x| + c$$

(b) $\boxed{v = 5t + 5}$.
The steps are
$v - \frac{1}{2(1+t}\,v = \frac{5}{2}$, standard linear form,
$\frac{(Wv)'}{W} = \frac{5}{2}$, $W =$ integrating factor $= e^{-\frac{1}{2}\ln|1+t|}$.
$(Wv)' = \frac{5}{2}W$, where $W = (1+t)^{-1/2}$ is the reduced form of W,
$Wv = \frac{5}{2}(1+t)^{1/2}\frac{1}{1/2} + c$, after a quadrature,
$v = \frac{5}{2}\frac{2}{1}(1+t)^{1/2+1/2} + c\,(1+t)^{1/2}$,
Now use $v(0) = 5$ to obtain from the above formula at $t = 0$ the relation
$5 = \frac{5}{2}\frac{2}{1}(1+0)^{1/2+1/2} + c\,(1+0)^{1/2}$ or $c = 0$.
Then $v(t) = 5t + 5$.
(c) $y(x) = c\,e^{\sqrt{1+x}}$ = constant divided by the integrating factor.
(d) $y_h(x) = Ce^{\frac{2}{\pi}\sqrt{x+1}}$ = constant divided by the integrating factor, $y_p(x) = 5\pi =$ equilibrium solution.

Chapter 3: Linear Equations of Higher Order

(a) [10%] Solve for the general solution: $y'' + 4y' + 5y = 0$

(b) [20%] Solve for the general solution: $y^{(6)} + 9y^{(4)} = 0$

(c) [20%] Solve for the general solution, given the characteristic equation is $r(r^3 + r)^2(r^2 + 2r + 17)^2 = 0$.

(d) [20%] Given $6x''(t) + 2x'(t) + 2x(t) = 11\cos(\omega t)$, which represents a damped forced spring-mass system with $m = 6$, $c = 2$, $k = 2$, answer the following questions.

True ☐ or False ☐. Practical mechanical resonance occurs for input frequency $\omega = \sqrt{11/6}$.

True ☐ or False ☐. The homogeneous problem is over-damped.

Answer:
(a) $r^2 + 4r + 5 = 0$, $y = c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x)$.
(b) $r^6 + 16r^4 = 0$, roots $r = 0, 0, 0, 0;\ 4i, -4i$. Then the Euler atoms are $1, x, x^2, x^3$; $\cos 4x, \sin 4x$. The general solution is a linear combination of the atoms.
(c) Write as $r^3(r + 1)^3(r - 1)^2((r + 1)^2 + 4)^2 = 0$. Then y is a linear combination of the Euler atoms 1, x, x^2, e^{-x}, xe^{-x}, x^2e^{-x}, e^x, xe^x, $e^{-x}\cos 2x$, $xe^{-x}\cos 2x$, $\sin 2x$, $xe^{-x}\sin 2x$.
(d) False and False. The resonant frequency is $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$. Use $6r^2 + 2r + 2 = 0$ and the quadratic formula to obtain complex conjugate roots. It is under-damped.

(e) [30%] Determine for $y^{(5)} + 4y^{(3)} = x^2 + e^x + \sin(2x)$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Answer:
(e) The homogeneous solution is a linear combination of the atoms 1, x, x^2, $\cos 2x$, $\sin 2x$ because the characteristic polynomial has roots 0, 0, 0, $2i$, $-2i$.
[1] An initial trial solution y is constructed by Rule I for atoms 1, x, x^2, e^x, $\cos 2x$, $\sin 2x$, $x\cos 2x$, $x\sin 2x$, giving

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 + d_2 x + d_3 x^2, \\ y_2 &= d_4 e^x, \\ y_3 &= d_5 \cos 2x + d_6 x \cos 2x, \\ y_4 &= d_7 \sin 2x + d_8 x \sin 2x. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by specialization of constants, all derivatives of the Euler atoms appearing in the right side of the differential equation.

2 Rule II is applied individually to each of y_1, y_2, y_3, y_4.

The result is the **shortest trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= x^3 d_1 + d_2 x^4 + d_3 x^5, \\ y_2 &= d_4 e^x, \\ y_3 &= d_5 x \cos 2x + d_6 x^2 \cos 2x, \\ y_4 &= d_7 x \sin 2x + d_8 x^2 \sin 2x. \end{aligned}$$

Chapters 4 and 5: Systems of Differential Equations

(a) [10%] Matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$ has eigenpairs

$$\left(-1, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(-5, \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}\right).$$

Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$. ______________

Answer:

(a): The eigenpairs are

$$\left(-1, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(-5, \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}\right).$$

An expected detail is the cofactor expansion of $\det(A - \lambda I)$ and factoring to find eigenvalues $-1, 1, -5$. Eigenvectors should be found by a sequence of swap, combo, mult operations on the augmented matrix, followed by taking the partial ∂_{t_1} on invented symbol t_1 in the general solution to compute the eigenvector. In short, the eigenvectors are Strang's Special Solutions. In general there can be many eigenvectors for a single eigenvalue, however, for distinct eigenvalues there is exactly one eigenvector per eigenvalue.

(b): The eigenanalysis method for $\mathbf{x}' = A\mathbf{x}$ implies

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}.$$

(b) [30%] Find the general solution of the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

according to the Cayley-Hamilton-Ziebur Method, using the textbook's shortcut in chapter 4. ______________

Answer:

(c) **Eigenvalue Calculation**

Subtract λ from the diagonal elements of $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ to obtain matrix $B = A - \lambda I$, then expand $\det(B)$ to obtain the characteristic polynomial. The roots are the eigenvalues $\lambda = 3, 5$.

Cayley-Hamilton-Ziebur Method

The eigenvalues $3, 5$ are used to create the list of atoms e^{3t}, e^{5t}. Then the Cayley-Hamilton-Ziebur method implies there are constants c_1, c_2 such that $x(t) = c_1 e^{3t} + c_2 e^{5t}$. Then the first differential equation $x' = 4x + y$ is solved for $y = x' - 4x$. Expand this equation using $x(t) = c_1 e^{3t} + c_2 e^{5t}$ to obtain $y(t) = x' - 4x = -c_1 e^{3t} + c_2 e^{5t}$.

(c) [10%] Assume a 2×2 system $\frac{d}{dt}\vec{u} = A\vec{u}$ has a scalar general solution

$$x(t) = c_1 e^{-t} + c_2 e^{4t}, \quad y(t) = 4c_2 e^{-t} + (c_1 - 2c_2)e^{4t}.$$

Compute a fundamental matrix $\Phi(t)$. ________________

Answer:

Fundamental Matrix.

Compute the partial derivatives $\partial/\partial c_1$, $\partial/\partial c_2$ to determine columns 1 and 2 of the 2×2 fundamental matrix $\Phi(t)$, using the answer given. Then

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{4t} \\ e^{4t} & 3e^{-t} - e^{4t} \end{pmatrix}.$$

(d) [20%] Consider the scalar system

$$\begin{cases} x' &= x \\ y' &= 3x, \\ z' &= x + y \end{cases}$$

Solve the system by the most efficient method. ________________

Answer:

Linear Cascade Method.

Solve the first equation by $x =$ constant divided by the integrating factor. Then $\boxed{x(t) = c_1 e^{3t}}$.

Substitute this formula into the second equation $y' = x$ and apply quadrature to obtain
$y' = c_1 e^{3t}$
$\boxed{y = \frac{1}{3} c_1 e^{3t} + c_2}$.

Substitute both x and y into the third equation $z' = x + y$ and again apply quadrature to obtain
$z' = \frac{4}{3} c_1 e^{3t} + c_2$
$\boxed{z = \frac{4}{9} c_1 e^{3t} + c_2 t + c_3}$.

The matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ is not diagonalizable, so only methods Laplace, Linear Cascade and Cayley-Hamilton-Ziebur can apply.

Chapter 6: Dynamical Systems

(a) [10%] The origin is an equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u}$. Classify $(0,0)$ as *center, spiral, node, saddle.*

Answer:
(a) All except the center and the saddle. The center is stable but not asymptotically stable. All the others correspond to a general solution which can have an exponential factor e^{kt} in each term. If $k < 0$, then the exponential limits to zero at $t = \infty$.
(b) Yes, because $\det(A) \neq 0$. In this case, $A\mathbf{u} = \mathbf{0}$ has a unique solution.

In parts (b), (c), (d), consider the nonlinear dynamical system

$$x' = 14x - 2x^2 - xy, \quad y' = 16y - 2y^2 - xy. \qquad (1)$$

(b) [20%] Find the equilibrium points for the nonlinear system (1).

(c) [30%] Consider again system (1). Classify the linearization at equilibrium point $(4,6)$ as a node, spiral, center, saddle.

(d) [30%] Consider again system (1). What classification can be deduced for equilibrium $(4,6)$ of this nonlinear system, according to the Pasting Theorem?

Answer:
The equilibria are constant solutions, which are found from the equations

$$\begin{aligned} 0 &= (14 - 2x - y)x \\ 0 &= (16 - 2y - x)y \end{aligned}$$

Considering when a zero factor can occur leads to the four equilibria $(0,0)$, $(0,8)$, $(7,0)$, $(4,6)$. The last equilibrium comes from solving the system of equations

$$\begin{aligned} 2x + y &= 14 \\ x + 2y &= 16 \end{aligned}$$

Linearization
The Jacobian matrix J is the augmented matrix of partial derivatives $\partial_x \vec{\mathbf{F}}$, $\partial_y \vec{\mathbf{F}}$ (column vectors) computed from

$$\vec{\mathbf{f}}(x,y) = \begin{pmatrix} 14x - 2x^2 - yx \\ 16y - 2y^2 - xy \end{pmatrix}.$$

Then

$$J(x,y) = \begin{pmatrix} 14-4x-y & -x \\ -y & 16-4y-x \end{pmatrix}.$$

The four matrices below are $J(x,y)$ when (x,y) is replaced by an equilibrium point. Included in the table are the roots of the characteristic equation for each matrix and its classification based on the roots. No book was consulted for the classifications. The idea in each is to examine the limits at $t=\pm\infty$, then eliminate classifications. No matrix has complex eigenvalues, and that eliminates the center and spiral. The first three are stable at either $t=\infty$ or $t=-\infty$, which eliminates the saddle and leaves the node as the only possible classification.

$A_1 = J(0,0) = \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix}$	$r=14,16$	node
$A_2 = J(0,8) = \begin{pmatrix} 6 & 0 \\ -8 & -16 \end{pmatrix}$	$r=6,-16$	saddle
$A_3 = J(7,0) = \begin{pmatrix} -14 & -7 \\ 0 & 9 \end{pmatrix}$	$r=-14,9$	saddle
$A_4 = J(4,6) = \begin{pmatrix} -8 & -4 \\ -6 & -12 \end{pmatrix}$	$r=-10-2\sqrt{7}, -10+2\sqrt{7}$	node

The pasting theorem says that a linearized saddle maps to a nonlinear saddle. In the present example, each node has unequal eigenvalues, and then the pasting theorem says that the linearized node maps to a nonlinear node. The stability is inherited for the saddle and the node.

Some maple code for checking the answers:

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F:=unapply([14*x-2*x^2-y*x , 16*y-2*y^2 -x*y],(x,y));
Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
Fx(0,0);Fy(0,0);Fx(7,0);Fy(7,0);Fx(0,8);Fy(0,8);Fx(4,6);Fy(4,6);
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Chapter 7: Laplace Theory

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Answer:
(a) Laplace's method explained.
The first step transforms the equation using the parts formula and initial data to get

$$(s^2 + 2s)\mathcal{L}(x) = \mathcal{L}(e^t).$$

The forward Laplace table applies to write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1}{(s-1)(s^2+2s)} = \frac{1}{s(s-1)(s+2)}.$$

Partial fraction methods imply

$$\mathcal{L}(x) = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+2} = \mathcal{L}(a + be^t + ce^{-2t})$$

and then $x(t) = a + be^t + ce^{-2t}$ by Lerch's theorem.
Partial Fractions.
The cleared fraction equation is

$$1 = a(s-1)(s+2) + bs(s+2) + cs(s-1).$$

Put $s = 0, 1, -2$ (roots of the denominator in the original fraction) into this equation to obtain three equations in three unknowns:

$$\begin{aligned} 1 &= -2a + 0 + 0, \\ 1 &= 0 + 3b + 0, \\ 1 &= 0 + 0 + 6c. \end{aligned}$$

The constants are $a = -1/2$, $b = 1/3$, $c = 1/6$.

(a) [10%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s(s+1)^2}$.

(b) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)\sinh(3t)$. This is the hyperbolic sine.

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Answer:
(b) $f(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}$.
Details.
$\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s+2} + \frac{c}{(s+2)^2} = \mathcal{L}(a + be^{-2t} + cte^{-2t})$ implies $f(t) = a + be^{-2t} + cte^{-2t}$. The third term was done by the First Shifting Theorem. The constants are found by clearing fractions to obtain the equation

$$1 = a(s+2)^2 + bs(s+2) + cs.$$

Put $s = 0, -2, 1$ to obtain a 3×3 system

$$\begin{array}{rcl} 1 &=& 4a + 0 + 0, \\ 1 &=& 0 + 0 - 2c, \\ 1 &=& 9a + 3b + c. \end{array}$$

The answer is $a = \frac{1}{4}$, $b = -\frac{1}{4}$, $c = -\frac{1}{2}$.

(c) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ by the s-differentiation theorem. The first shifting theorem and the forward table imply

$$\mathcal{L}(e^{2t}\sin 3t) = \mathcal{L}(\sin 3t)|_{s\to(s-2)} = \left.\frac{3}{s^2+9}\right|_{s\to(s-2)}.$$

Finally,

$$\mathcal{L}(f) = \frac{d}{ds}\left(\frac{3}{(s-2)^2+9}\right) = \frac{-6(s-2)}{((s-2)^2+9)^2}.$$

(c) [30%] Solve by Laplace's Method the forced linear dynamical system

$$\begin{cases} x' &=& x - y + 2, \\ y' &=& x + y + 1, \end{cases}$$

subject to initial states $x(0) = 0$, $y(0) = 0$.

Answer:

(d) Transform to get the two equations

$$\begin{cases} (s-1)\mathcal{L}(x) &+& (1)\mathcal{L}(y) &=& 0, \\ (-1)\mathcal{L}(x) &+& (s-1)\mathcal{L}(y) &=& \frac{2}{s-1}. \end{cases}$$

Solve with Cramer's Rule to obtain

$\mathcal{L}(x) = \left(\frac{-2}{s-1}\right)\frac{1}{(s-1)^2+1}$,

$\mathcal{L}(y) = \frac{2}{(s-1)^2+1}$.

The first requires partial fraction decomposition. Details:

$\mathcal{L}(x) = \frac{-2}{s-1}\frac{1}{(s-1)^2+1}$,

$\mathcal{L}(x) = \left.\frac{-2}{s}\frac{1}{s^2+1}\right|_{s\to(s-1)}$

$\mathcal{L}(x) = \left.\frac{a}{s} + \frac{bs+c}{s^2+1}\right|_{s\to(s-1)}$,

$\mathcal{L}(x) = \mathcal{L}(a + b\cos t + c\sin t)|_{s\to(s-1)}$,

$\mathcal{L}(x) = \mathcal{L}((a + b\cos t + c\sin t)e^t)$.

Then $x(t) = (a + b\cos t + c\sin t)e^t$. The constants are $a = -2, b = 2, c = 0$.

$\boxed{x(t) = (-2 + 2\cos t)e^t}$

The second equation is finished with the first shifting theorem and the Laplace table,

$\mathcal{L}(y) = \frac{2}{(s-1)^2+1}$,
$\mathcal{L}(y) = \left.\frac{2}{(s^2+1}\right|_{s\to(s-1)}$,
$\mathcal{L}(y) = \mathcal{L}(2\sin t)|_{s\to(s-1)}$,
$\mathcal{L}(y) = \mathcal{L}(2e^t \sin t)$. Then
$\boxed{y(t) = 2e^t \sin t}$

(d) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s}{s^2+2s+17}$.

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left.\left(\mathcal{L}\left(t^2e^{4t}\cos t\right)\right)\right|_{s\to s+2}.$$

Answer:
(e) $f(t) = e^{-t}(\cos 4t - \sin 4t)$
Details: Factor the quadratic: $s^2 + 2s + 5 = (s+1)^2 + 4$. Then shift by $s \to (s+1)$ to obtain
$\mathcal{L}(f) = \left.\frac{s-2}{s^2+4}\right|_{s\to(s+1)}$
$\mathcal{L}(f) = (\mathcal{L}(\cos 4t) - \mathcal{L}(\sin 4t))|_{s\to(s+1)}$
$\mathcal{L}(f) = \mathcal{L}(e^{-t}(\cos 4t - \sin 4t))$
Then Lerch's theorem implies $f(t) = e^{-t}(\cos 4t - \sin 4t)$.
(f) $f(t) = t^3e^{6t}\cos 8t$
Details: The first shifting theorem $\mathcal{L}(g(t))|_{s\to(s-a)} = \mathcal{L}(e^{at}g(t))$ is applied to remove the shift on the outside and put e^{-3t} into the Laplace integrand. Then $\mathcal{L}(f(t)) = \mathcal{L}(e^{-3t}t^3e^{9t}\cos 8t)$. Lerch's theorem implies $f(t) = t^3e^{6t}\cos 8t$.

Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let $f_0(x) = 1$ on the interval $-1 < x < 0$, $f_0(x) = -1$ on the interval $0 < x < 1$, $f_0(x) = 0$ for $x = 0$ and $x = \pm 1$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 2.

(a) [10%] Compute the Fourier coefficients of $f(x)$ on $[-1, 1]$.

(b) [10%] Find all values of x in $|x| < 3$ which will exhibit Gibb's over-shoot.

Answer:
(a) Because $f(x)$ is odd, then $f(x)$ times a cosine is odd. Then the coefficient a_n of a cosine term is zero, because the integral of an odd function over $-1 < x < 1$ is zero.
(b) There is a jump discontinuity of $f(x)$ at $x = 1, 3, -1, -3$. At these points there is a Gibbs overshoot.

Answer:
(c) INTEGRATION.
Assume $f(x)$ is piecewise continuous and periodic of period $2L$ on the whole real line. Then

$$\int_0^t f(x)dx = \frac{a_0}{2}t + \frac{a_0}{2} + \sum_{n=1}^{\infty} \int_0^t (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L))\, dt$$

where the Fourier coefficients are given by the inner product formulas

$$a_n = \frac{< f(x), \cos(n\pi x/L) >}{< \cos(n\pi x/L), \cos(n\pi x/L) >} = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)dx,$$

$$b_n = \frac{< f(x), \sin(n\pi x/L) >}{< \sin(n\pi x/L), \sin(n\pi x/L) >} = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx.$$

(d) [40%] **Heat Conduction in a Rod**. Solve the rod problem on $0 \le x \le L$, $t \ge 0$:

$$\begin{cases} u_t &= u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= 5\sin(2\pi x/L) + 12\sin(4\pi x/L) \end{cases}$$

Answer: (d)

The solution uses trig identity $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$, as follows.
$u(x,0) = f(x) = 5\sin^2(2\pi x/L) = \frac{5}{2}(1 - \cos(4\pi x/L)))$

The known solution for temperature $u(x,t)$ is a Fourier sine series for $f(x)$ with exponentials inserted according to Fourier's method:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) e^{-n^2\pi^2 t/L^2}.$$

It remains to find the Fourier coefficients:

$b_n = \frac{<f(x), \sin(n\pi x/L)>}{<\sin(n\pi x/L), \sin(n\pi x/L)>}$

$= \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$

$= \frac{2}{L}\frac{5}{2}\int_0^L (1 - \cos(4\pi x/L))\sin(n\pi x/L)dx$

$= 80\dfrac{-1 + (-1)^n}{n\pi(n^2 - 16)}$ when $n \neq 4$ and

$= 0$ for $n = 4$. 5 80 1 + (-1) /

(e) [30%] **Vibration of a Finite String**. The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ on $0 < x < L$, $t > 0$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes (the superposition of the normal modes).

Solve the finite string vibration problem on $0 \le x \le 5$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &=& 25u_{xx}(x,t), \\ u(0,t) &=& 0, \\ u(5,t) &=& 0, \\ u(x,0) &=& \sin(5\pi x) + 2\sin(7\pi x), \\ u_t(x,0) &=& 0 \end{cases}$$

Answer: (e)
Because the wave velocity is zero, then the only normal modes are $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$.

Because the wave initial shape $f(x) = \sin(\pi x) + 5\sin(11\pi x)$ is already a sine series, then it suffices by Fourier's method to insert cosine factors, to create appropriate normal modes. Then $c^2 = 100$ and $L = 5$ implies

$u(x,t) = \sin(5\pi x/L)\cos(50\pi t/L) + 5\sin(55\pi x/L)\cos(550\pi t/L)$

$$= \sin(\pi x)\cos(10\pi t) + 5\sin(11\pi x)\cos(110\pi t).$$

We check it is a solution.