

**Differential Equations 2280**  
**Sample Midterm Exam 3 with Solutions**  
**Exam Date: Friday 22 April 2016 at 12:50pm**

**Instructions:** This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exams 1, 2.

### Chapter 3

1. (Linear Constant Equations of Order  $n$ )

(a) Find by variation of parameters a particular solution  $y_p$  for the equation  $y'' = 1 - x$ . Show all steps in variation of parameters. Check the answer by quadrature.

(b) A particular solution of the equation  $mx'' + cx' + kx = F_0 \cos(2t)$  happens to be  $x(t) = 11 \cos 2t + e^{-t} \sin \sqrt{11}t - \sqrt{11} \sin 2t$ . Assume  $m, c, k$  all positive. Find the unique periodic steady-state solution  $x_{ss}$ .

(c) A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions  $2e^{3x} + 4x$  and  $xe^{3x}$ . Write a formula for the general solution.

(d) Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40 \cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time, don't convert to phase-amplitude form.**

(e) Write the solution  $x(t)$  of

$$x''(t) + 25x(t) = 180 \sin(4t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.

**To save time, don't convert to phase-amplitude form.**

(f) Find the steady-state periodic solution for the forced spring-mass system  $x'' + 2x' + 2x = 5 \sin(t)$ .

(g) Given  $5x''(t) + 2x'(t) + 4x(t) = 0$ , which represents a damped spring-mass system with  $m = 5$ ,  $c = 2$ ,  $k = 4$ , determine if the equation is over-damped, critically damped or under-damped.

**To save time, do not solve for  $x(t)$ !**

(h) Determine the practical resonance frequency  $\omega$  for the electric current equation

$$2I'' + 7I' + 50I = 100\omega \cos(\omega t).$$

(i) Given the forced spring-mass system  $x'' + 2x' + 17x = 82 \sin(5t)$ , find the steady-state periodic solution.

(j) Let  $f(x) = x^3 e^{1.2x} + x^2 e^{-x} \sin(x)$ . Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has  $f(x)$  as a solution. To save time, do not expand the polynomial and do not find the differential equation.

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**Answers and Solution Details:**

**Part (a)** Answer:  $y_p = \frac{x^2}{2} - \frac{x^3}{6}$ .

**Variation of Parameters.**

Solve  $y'' = 0$  to get  $y_h = c_1 y_1 + c_2 y_2$ ,  $y_1 = 1$ ,  $y_2 = x$ . Compute the Wronskian  $W = y_1 y_2' - y_1' y_2 = 1$ . Then for  $f(t) = 1 - x$ ,

$$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx,$$

$$y_p = 1 \int -x(1-x) dx + x \int 1(1-x) dx,$$

$$y_p = -1(x^2/2 - x^3/3) + x(x - x^2/2),$$

$$y_p = x^2/2 - x^3/6.$$

This answer is checked by quadrature, applied twice to  $y'' = 1 - x$  with initial conditions zero.

**Part (b)** It has to be the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then  $x_{ss}(t) = 11 \cos 2t - \sqrt{11} \sin 2t$ .

**Part (c)** In order for  $x e^{3x}$  to be a solution, the general solution must have Euler atoms  $e^{3x}$ ,  $x e^{3x}$ . Then the first solution  $2e^{3x} + 4x$  minus 2 times the Euler atom  $e^{3x}$  must be a solution, therefore  $x$  is a solution, resulting in Euler atoms  $1, x$ . The general solution is then a linear combination of the four Euler atoms:  $y = c_1(1) + c_2(x) + c_3(e^{3x}) + c_4(xe^{3x})$ .

**Part (d)** Use undetermined coefficients trial solution  $x = d_1 \cos 4t + d_2 \sin 4t$ . Then  $d_1 = 5/6$ ,  $d_2 = 0$ , and finally  $x_p(t) = (5/6) \cos(4t)$ . The characteristic equation  $r^2 + 64 = 0$  has roots  $\pm 8i$  with corresponding Euler solution atoms  $\cos(8t), \sin(8t)$ . Then  $x_h(t) = c_1 \cos(8t) + c_2 \sin(8t)$ . The general solution is  $x = x_h + x_p$ . Now use  $x(0) = x'(0) = 0$  to determine  $c_1 = -5/6, c_2 = 0$ , which implies the particular solution  $x(t) = -\frac{5}{6} \cos(8t) + \frac{5}{6} \cos(4t)$ .

**Part (e)** The answer is  $x(t) = -16 \sin(5t) + 20 \sin(4t)$  by the method of undetermined coefficients.

Rule I:  $x = d_1 \cos(4t) + d_2 \sin(4t)$ . Rule II does not apply due to natural frequency  $\sqrt{25} = 5$  not equal to the frequency of the trial solution (no conflict). Substitute the trial solution into  $x''(t) + 25x(t) = 180 \sin(4t)$  to get  $9d_1 \cos(4t) + 9d_2 \sin(4t) = 180 \sin(4t)$ . Match coefficients, to arrive at the equations  $9d_1 = 0$ ,  $9d_2 = 180$ . Then  $d_1 = 0$ ,  $d_2 = 20$  and  $x_p(t) = 20 \sin(4t)$ . Lastly, add the homogeneous solution to obtain  $x(t) = x_h + x_p = c_1 \cos(5t) + c_2 \sin(5t) + 20 \sin(4t)$ . Use the initial condition relations  $x(0) = 0, x'(0) = 0$  to obtain the equations  $\cos(0)c_1 + \sin(0)c_2 + 20 \sin(0) = 0$ ,  $-5 \sin(0)c_1 + 5 \cos(0)c_2 + 80 \cos(0) = 0$ . Solve for the coefficients  $c_1 = 0, c_2 = -16$ .

**Part (f)** The answer is  $x = \sin t - 2 \cos t$  by the method of undetermined coefficients.

Rule I: the trial solution  $x(t)$  is a linear combination of the Euler atoms found in  $f(x) = 5 \sin(t)$ . Then  $x(t) = d_1 \cos(t) + d_2 \sin(t)$ . Rule II does not apply, because solutions of the homogeneous problem contain negative exponential factors (no conflict). Substitute the trial solution into  $x'' + 2x' + 2x = 5 \sin(t)$  to get  $(-2d_1 + d_2) \sin(t) + (d_1 + 2d_2) \cos(t) = 5 \sin(t)$ . Match coefficients to find the system of equations  $(-2d_1 + d_2) = 5, (d_1 + 2d_2) = 0$ . Solve for the coefficients  $d_1 = -2, d_2 = 1$ .

**Part (g)** Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is  $b^2 - 4ac = 2^2 - 4(5)(4) = (19)(-4)$ , therefore there are two complex conjugate roots and the equation is **under-damped**. Alternatively, factor  $5r^2 + 2r + 4$  to obtain roots  $(-1 \pm \sqrt{19}i)/5$  and then classify as **under-damped**.

**Part (h)** The resonant frequency is  $\omega = 1/\sqrt{LC} = 1/\sqrt{2/50} = \sqrt{25} = 5$ . The solution uses the theory in the textbook, section 3.7, which says that electrical resonance occurs for  $\omega = 1/\sqrt{LC}$ .

**Part (i)** The answer is  $x(t) = -5 \cos(5t) - 4 \sin(5t)$  by undetermined coefficients.

Rule I: The trial solution is  $x_p(t) = A \cos(5t) + B \sin(5t)$ . Rule II: because the homogeneous solution  $x_h(t)$  has limit zero at  $t = \infty$ , then Rule II does not apply (no conflict). Substitute the trial solution into the differential equation. Then  $-8A \cos(5t) - 8B \sin(5t) - 10A \sin(5t) + 10B \cos(5t) = 82 \sin(5t)$ . Matching coefficients of sine and cosine gives the equations  $-8A + 10B = 0$ ,  $-10A - 8B = 82$ . Solving,  $A = -5$ ,  $B = -4$ . Then  $x_p(t) = -5 \cos(5t) - 4 \sin(5t)$  is the unique periodic steady-state solution.

**Part (j)** The characteristic polynomial is the expansion  $(r - 1.2)^4((r + 1)^2 + 1)^3$ . Because  $x^3 e^{ax}$  is an Euler solution atom for the differential equation if and only if  $e^{ax}$ ,  $x e^{ax}$ ,  $x^2 e^{ax}$ ,  $x^3 e^{ax}$  are Euler solution atoms, then the characteristic equation must have roots 1.2, 1.2, 1.2, 1.2, listing according to multiplicity. Similarly,  $x^2 e^{-x} \sin(x)$  is an Euler solution atom for the differential equation if and only if  $-1 \pm i$ ,  $-1 \pm i$ ,  $-1 \pm i$  are roots of the characteristic equation. There is a total of 10 roots with product of the factors  $(r - 1)^4((r + 1)^2 + 1)^3$  equal to the 10th degree characteristic polynomial.

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Use this page to start your solution.

## Chapters 4 and 5

### 2. (Systems of Differential Equations)

**Background.** Let  $A$  be a real  $3 \times 3$  matrix with eigenpairs  $(\lambda_1, \mathbf{v}_1)$ ,  $(\lambda_2, \mathbf{v}_2)$ ,  $(\lambda_3, \mathbf{v}_3)$ . The eigenanalysis method says that the  $3 \times 3$  system  $\mathbf{x}' = A\mathbf{x}$  has general solution

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + c_3 \mathbf{v}_3 e^{\lambda_3 t}.$$

**Background.** Let  $A$  be an  $n \times n$  real matrix. The method called **Cayley-Hamilton-Ziebur** is based upon the result

The components of solution  $\mathbf{x}$  of  $\mathbf{x}'(t) = A\mathbf{x}(t)$  are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation  $|A - \lambda I| = 0$ .

**Background.** Let  $A$  be an  $n \times n$  real matrix. An augmented matrix  $\Phi(t)$  of  $n$  independent solutions of  $\mathbf{x}'(t) = A\mathbf{x}(t)$  is called a **fundamental matrix**. It is known that the general solution is  $\mathbf{x}(t) = \Phi(t)\mathbf{c}$ , where  $\mathbf{c}$  is a column vector of arbitrary constants  $c_1, \dots, c_n$ . An alternate and widely used definition of fundamental matrix is  $\Phi'(t) = A\Phi(t)$ ,  $|\Phi(0)| \neq 0$ .

(a) Display eigenanalysis details for the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

then display the general solution  $\mathbf{x}(t)$  of  $\mathbf{x}'(t) = A\mathbf{x}(t)$ .

(b) The  $3 \times 3$  triangular matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models. Using the linear integrating factor method, starting with component  $x_3(t)$ , find the vector general solution  $\mathbf{x}(t)$  of  $\mathbf{x}'(t) = A\mathbf{x}(t)$ .

(c) The exponential matrix  $e^{At}$  is defined to be a fundamental matrix  $\Psi(t)$  selected such that  $\Psi(0) = I$ , the  $n \times n$  identity matrix. Justify the formula  $e^{At} = \Phi(t)\Phi(0)^{-1}$ , valid for *any* fundamental matrix  $\Phi(t)$ .

(d) Let  $A$  denote a  $2 \times 2$  matrix. Assume  $\mathbf{x}'(t) = A\mathbf{x}(t)$  has scalar general solution  $x_1 = c_1 e^t + c_2 e^{2t}$ ,  $x_2 = (c_1 - c_2)e^t + 2c_1 + c_2)e^{2t}$ , where  $c_1, c_2$  are arbitrary constants. Find a fundamental matrix  $\Phi(t)$  and then go on to find  $e^{At}$  from the formula in part (c) above.

(e) Let  $A$  denote a  $2 \times 2$  matrix and consider the system  $\mathbf{x}'(t) = A\mathbf{x}(t)$ . Assume fundamental matrix  $\Phi(t) = \begin{pmatrix} e^t & e^{2t} \\ 2e^t & -e^{2t} \end{pmatrix}$ . Find the  $2 \times 2$  matrix  $A$ .

(f) The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 3x + y, \quad y' = -x + 3y,$$

which has complex eigenvalues  $\lambda = 3 \pm i$ . Show the details of the method, then go on to report a fundamental matrix  $\Phi(t)$ .

**Remark.** The vector general solution is  $\mathbf{x}(t) = \Phi(t)\mathbf{c}$ , which contains no complex numbers. Reference: 4.1, Examples 6,7,8.

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**Answers and Solution Details:**

**Part (a)** The details should solve the equation  $|A - \lambda I| = 0$  for the three eigenvalues  $\lambda = 5, 4, 3$ . Then solve the three systems  $(A - \lambda I)\vec{v} = \vec{0}$  for eigenvector  $\vec{v}$ , for  $\lambda = 5, 4, 3$ .

The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

**Part (b)** Write the system in scalar form

$$\begin{aligned} x' &= 3x + y + z, \\ y' &= 4y + z, \\ z' &= 5z. \end{aligned}$$

Solve the last equation as

$$z = \frac{\text{constant}}{\text{integrating factor}} = c_3 e^{5t}.$$

$$\boxed{z = c_3 e^{5t}}$$

The second equation is

$$y' = 4y + c_3 e^{5t}$$

The linear integrating factor method applies.

$$y' - 4y = c_3 e^{-5t}$$

$$\frac{(Wy)'}{W} = c_3 e^{5t}, \text{ where } W = e^{-4t},$$

$$(Wy)' = c_3 W e^{5t}$$

$$(e^{-4t}y)' = c_3 e^{-4t} e^{5t}$$

$$e^{-4t}y = c_3 e^t + c_2.$$

$$\boxed{y = c_3 e^{5t} + c_2 e^{4t}}$$

Stuff these two expressions into the first differential equation:

$$x' = 3x + y + z = 3x + 2c_3 e^{5t} + c_2 e^{4t}$$

Then solve with the linear integrating factor method.

$$x' - 3x = 2c_3 e^{5t} + c_2 e^{4t}$$

$$\frac{(Wx)'}{W} = 2c_3 e^{5t} + c_2 e^{4t}, \text{ where } W = e^{-3t}. \text{ Cross-multiply:}$$

$$(e^{-3t}x)' = 2c_3 e^{5t} e^{-3t} + c_2 e^{4t} e^{-3t}, \text{ then integrate:}$$

$$e^{-3t}x = c_3 e^{2t} + c_2 e^t + c_1$$

$$e^{-3t}x = c_3 e^{2t} + c_2 e^t + c_1, \text{ divide by } e^{-3t}:$$

$$\boxed{x = c_3 e^{5t} + c_2 e^{4t} + c_1 e^{3t}}$$

**Part (c)** The question reduces to showing that  $e^{At}$  and  $\Phi(t)\Phi(0)^{-1}$  have equal columns. This is done by showing that the matching columns are solutions of  $\vec{u}' = A\vec{u}$  with the same initial condition  $\vec{u}(0)$ , then apply Picard's theorem on uniqueness of initial value problems.

**Part (d)** Take partial derivatives on the symbols  $c_1, c_2$  to find vector solutions  $\vec{v}_1(t), \vec{v}_2(t)$ . Define  $\Phi(t)$  to be the augmented matrix of  $\vec{v}_1(t), \vec{v}_2(t)$ . Compute  $\Phi(0)^{-1}$ , then multiply on the right of  $\Phi(t)$  to obtain

$e^{At} = \Phi(t)\Phi(0)^{-1}$ . Check the answer in a computer algebra system or using Putzer's formula.

**Part (e)** The equation  $\Phi'(t) = A\Phi(t)$  holds for every  $t$ . Choose  $t = 0$  and then solve for  $A = \Phi'(0)\Phi(0)^{-1}$ .

**Part (f)** By C-H-Z,  $x = c_1e^{3t}\cos(t) + c_2e^{3t}\sin(t)$ . Isolate  $y$  from the first differential equation  $x' = 3x + y$ , obtaining the formula  $y = x' - 3x = -c_1e^{3t}\sin(t) + c_2e^{3t}\cos(t)$ . A fundamental matrix is found by taking partial derivatives on the symbols  $c_1, c_2$ . The answer is exactly the Jacobian matrix of  $\begin{pmatrix} x \\ y \end{pmatrix}$  with respect to variables  $c_1, c_2$ .

$$\Phi(t) = \begin{pmatrix} e^{3t}\cos(t) & e^{3t}\sin(t) \\ -e^{3t}\sin(t) & e^{3t}\cos(t) \end{pmatrix}.$$

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Use this page to start your solution.

## Chapter 6

## 3. (Linear and Nonlinear Dynamical Systems)

(a) Determine whether the unique equilibrium  $\vec{u} = \vec{0}$  is stable or unstable. Then classify the equilibrium point  $\vec{u} = \vec{0}$  as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

(b) Determine whether the unique equilibrium  $\vec{u} = \vec{0}$  is stable or unstable. Then classify the equilibrium point  $\vec{u} = \vec{0}$  as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{u}$$

(c) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - y + 32, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is  $x = 4, y = 4$ . Compute the Jacobian matrix  $A = J(4, 4)$  of the linearized system at this equilibrium point.

(d) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -x - 2y^2 - y + 32, \\ y' &= 2x^2 + 2xy. \end{aligned}$$

An equilibrium point is  $x = -4, y = 4$ . Compute the Jacobian matrix  $A = J(-4, 4)$  of the linearized system at this equilibrium point.

(e) Consider the nonlinear dynamical system  $\begin{cases} x' = -4x + 4y + 9 - x^2, \\ y' = 3x - 3y. \end{cases}$

At equilibrium point  $x = 3, y = 3$ , the Jacobian matrix is  $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$ .

(1) Determine the stability at  $t = \infty$  and the phase portrait classification saddle, center, spiral or node at  $\vec{u} = \vec{0}$  for the linear system  $\frac{d}{dt}\vec{u} = A\vec{u}$ .

(2) Apply the Pasting Theorem to classify  $x = 3, y = 3$  as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(f) Consider the nonlinear dynamical system  $\begin{cases} x' = -4x - 4y + 9 - x^2, \\ y' = 3x + 3y. \end{cases}$

At equilibrium point  $x = 3, y = -3$ , the Jacobian matrix is  $A = J(3, -3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$ .

**Linearization.** Determine the stability at  $t = \infty$  and the phase portrait classification saddle, center, spiral or node at  $\vec{u} = \vec{0}$  for the **linear dynamical system**  $\frac{d}{dt}\vec{u} = A\vec{u}$ .

**Nonlinear System.** Apply the Pasting Theorem to classify  $x = 3, y = -3$  as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

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**Answers and Solution Details:**

**Part (a)** It is an unstable spiral. Details: The eigenvalues of  $A$  are roots of  $r^2 - 2r + 5 = (r - 1)^2 + 4 = 0$ , which are complex conjugate roots  $1 \pm 2i$ . Rotation eliminates the saddle and node. Finally, the atoms  $e^t \cos 2t$ ,  $e^t \sin 2t$  have limit zero at  $t = -\infty$ , therefore the system is stable at  $t = -\infty$  and unstable at  $t = \infty$ . So it must be a spiral [centers have no exponentials]. Report: **unstable spiral**.

**Part (b)** It is a stable spiral. Details: The eigenvalues of  $A$  are roots of  $r^2 + 2r + 5 = (r + 1)^2 + 4 = 0$ , which are complex conjugate roots  $-1 \pm 2i$ . Rotation eliminates the saddle and node. Finally, the atoms  $e^{-t} \cos 2t$ ,  $e^{-t} \sin 2t$  have limit zero at  $t = \infty$ , therefore the system is stable at  $t = \infty$  and unstable at  $t = -\infty$ . So it must be a spiral [centers have no exponentials]. Report: **stable spiral**.

**Part (c)** The Jacobian is  $J(x, y) = \begin{pmatrix} 1 & -4y - 1 \\ 4x - 2y & -2x \end{pmatrix}$ . Then  $A = J(4, 4) = \begin{pmatrix} 1 & -17 \\ 8 & -8 \end{pmatrix}$ .

**Part (d)** The Jacobian is  $J(x, y) = \begin{pmatrix} -1 & -4y - 1 \\ 4x + 2y & 2x \end{pmatrix}$ . Then  $A = J(-4, 4) = \begin{pmatrix} -1 & -17 \\ -8 & -8 \end{pmatrix}$ .

**Part (e)** (1) The Jacobian is  $J(x, y) = \begin{pmatrix} -4 - 2x & 4 \\ 3 & -3 \end{pmatrix}$ . Then  $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$ . The eigenvalues of  $A$  are found from  $r^2 + 13r + 18 = 0$ , giving distinct real negative roots  $-\frac{13}{2} \pm (\frac{1}{2})\sqrt{97}$ . Because there are no trig functions in the Euler solution atoms, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms limit to zero at  $t = \infty$ , therefore it is a node and we report a **stable node** for the linear problem  $\vec{u}' = A\vec{u}$  at equilibrium  $\vec{u} = \vec{0}$ .

(2) Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: **stable node** at  $x = 3$ ,  $y = 3$ . The exceptional case in Theorem 2 is a proper node, having characteristic equation roots that are equal. Stability is always preserved for nodes.

**Part (f)**

**Linearization.** The Jacobian is  $J(x, y) = \begin{pmatrix} -4 - 2x & -4 \\ 3 & 3 \end{pmatrix}$ . Then  $A = J(3, 3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$ . The

eigenvalues of  $A$  are found from  $r^2 + 7r - 18 = 0$ , giving distinct real roots  $2, -9$ . Because there are no trig functions in the Euler solution atoms  $e^{2t}, e^{-9t}$ , then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms do not limit to zero at  $t = \infty$  or  $t = -\infty$ , therefore it is a saddle and we report a **unstable saddle** for the linear problem  $\vec{u}' = A\vec{u}$  at equilibrium  $\vec{u} = \vec{0}$ .

**Nonlinear System.** Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: **unstable saddle** at  $x = 3$ ,  $y = 3$ .

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**Final Exam Problems**

**Chapter 5.** Solve a homogeneous system  $u' = Au$ ,  $u(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ,  $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$  using the matrix exponential, Zeibur's method, Laplace resolvent and eigenanalysis.

**Chapter 5.** Solve a non-homogeneous system  $u' = Au + F(t)$ ,  $u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ ,  $F(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  using variation of parameters.