Name ________________________________

Differential Equations 2280

Midterm Exam 2

Exam Date: 1 April 2016 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 1)**

(a) [30%] Solve $y' + 2y = 3$.

(b) [30%] Solve $y' + 2xy = 0$.

(c) [40%] Solve $y' + y = 2e^x$.

Answer:

(a) Solution by superposition shortcut for constant cefficient equations: $y_h = c/e^{2x}$, $y_p = 3/2$, $y = y_h + y_p$.

(b) Solution by homogeneous equation shortcut: $y = c/e^{x^2}$.

(c) Solution by the integrating factor method: $Q = e^x$ is the integrating factor. Then $(yQ)'/Q = 2e^x$ implies $yQ = \int 2Qe^x dx = \int 2e^{2x} dx = e^{2x} + c$. The answer is $y = e^x + c/e^x$.

Start your solution on this page.

2. **(Chapter 3)**

(a) [30%] Find the factors of the characteristic equation of a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution

$$y(x) = 10 + 5xe^x \sin(x) + xe^{-x}.$$

(b) [40%] Determine for differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3 + e^{-x} + \cos x$$

the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

(c) [30%] Find the steady-state periodic solution for the spring-mass equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 17x = 130\cos(t),$$

given a particular solution

$$x(t) = 4e^{-t}\sin(4\,t) + 5e^{-t}\cos(4\,t) + \sin(t) + 8\,\cos(t)$$

Answer:

(a) The atoms that appear in $y(x) = 10 + 5xe^x \sin(x) + xe^{-x}$ are $1, xe^x \sin x, xe^{-x}$. Derivatives of these atoms create a longer list: $1, e^x \cos x, e^x \sin x, xe^x \cos x, xe^x \sin x, e^{-x}, xe^{-x}$. These atoms correspond to characteristic equation roots 0; $1+i$, $1-i$, $1+i$, $1-i$; -1, -1. Then the characteristic equation has factors r; $((r-1)^2+1)^2$; $(r+1)^2$. The product of these factors is the characteristic equation which corresponds to the differential equation of least order such that $y(x)$ is a solution.

(b) The homogeneous solution is a linear combination of the atoms 1, e^{-x} because the characteristic polynomial has roots 0, -1.

Rule $\boxed{1}$ An initial trial solution y is constructed for the atoms of $f(x) = x^3 + e^{-x} + \cos x$ and its derivatives. The list of Euler solution atoms is 1, x, x^2, x^3, e^{-x}, $\cos x$, $\sin x$ giving 4 groups, each group having the same base atom:

$$\begin{array}{rcl} y & = & y_1 + y_2, \\ y_1 & = & d_1 + d_2 x + d_3 x^2 + d_4 x^3, \\ y_2 & = & d_5 e^{-x}, \\ y_3 & = & d_6 \cos x, \\ y_4 & = & d_7 \sin x. \end{array}$$

Linear combinations of the listed independent Euler solution atoms are supposed to reproduce, by specialization of constants, all derivatives of the right side of the differential equation.

Rule $\boxed{2}$ The correction rule is applied individually to each of y_1, y_2, y_3, y_4.
Multiplication by x is done repeatedly, until the replacement atoms do not appear in atom list for the homogeneous differential equation. The result is the **shortest trial solution**

$$y = y_1 + y_2 = (d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4) + (d_5 x e^{-x}) + (d_6 \cos x) + (d_7 \sin x).$$

The general solution of the differential equation according to Maple is

$$y(x) = 1/4\,x^4 - x^3 + 3\,x^2 + 1/2\,\sin(x) - 1/2\,\cos(x) - \mathrm{e}^{-x}x - \mathrm{e}^{-x} - 6\,x + c_1\mathrm{e}^{-x} + c_2.$$

(c) The answer is $x_{SS}(t) = 8\cos t + \sin t$. This result is obtained by cross-out of the terms that limit to zero at infinity, namely, all terms with a factor of e^{-t}. This finishes the solution steps. Included below is the solution by the method of undetermined coefficients, for the few instances in which that path was taken to solve the problem (not recommended).

Undetermined Coefficient Details. The trial solution by Rule I is $x(t) = d_1 \cos t + d_2 \sin t$. The homogeneous solutions have exponential factors, therefore the Euler atoms in the trial solution cannot be solutions of the homogeneous problem, hence Rule II does not apply.

Substitute the trial solution into the non-homogeneous equation to obtain the answers $d_1 = 8$, $d_2 = 1$ (steps omitted here). The unique periodic solution x_{SS} is extracted from the general solution $x = x_h + x_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). Because $x_p = d_1 \cos t + d_2 \sin t = 8\cos t + \sin t$ and the homogeneous solution x_h has negative exponential terms, then

$$x_{SS} = 8\cos t + \sin t.$$

Start your solution on this page.

3. **(Laplace Theory)**

(a) [50%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\left(\frac{d}{ds}\mathcal{L}(f(t))\right)\Big|_{s\to(s+5)} = \frac{1}{s^2} + \frac{1}{(s+2)^2}.$$

(b) [50%] Find $\mathcal{L}(f)$ given $f(t) = e^{2t}\sin(t) + (e^t + e^{-t})^2$.

Answer:

(a) Replace by the shift theorem and the s-differentiation theorem the given equation by

$$\mathcal{L}\left((-t)f(t)e^{-5t}\right) = \frac{1}{s^2} + \frac{1}{(s+2)^2}.$$

Then

$$\mathcal{L}\left((-t)f(t)e^{-5t}\right) = \mathcal{L}(t) + \mathcal{L}(te^{-2t}).$$

Then Lerch's theorem cancels $\mathcal{L}$ to give $(-t)e^{-5t}f(t) = t + te^{-2t}$. Solve for

$$f(t) = -\frac{t + te^{-2t}}{te^{-5t}}.$$

Then

$$f(t) = -e^{5t} - e^{3t}.$$

(b) Write $f = f_1 + f_2$ where $f_1(t) = e^{2t}\sin(t)$ and $f_2(t) = (e^t + e^{-t})^2 = e^{2t} + 2 + e^{-2t}$. Then the first shifting theorem implies

$$\begin{aligned}\mathcal{L}(f_1) &= \mathcal{L}(\sin t)\big|_{s:=s-2} \\ &= \frac{1}{s^2+1}\Big|_{s:=s-2} \\ &= \frac{1}{(s-2)^2+1}.\end{aligned}$$

The details for f_2:

$$\mathcal{L}(f_2(t)) = \mathcal{L}(e^{2t} + \mathcal{L}(2) + \mathcal{L}(e^{-2t}) = \frac{1}{s-2} + \frac{2}{s} + \frac{1}{s+2}.$$

The

$$\mathcal{L}(f(t)) = \mathcal{L}(f_1(t)) + \mathcal{L}(f_2(t)) = \frac{1}{(s-2)^2+1} + \frac{1}{s-2} + \frac{2}{s} + \frac{1}{s+2}.$$

Start your solution on this page.

4. **(Laplace Theory)**

(a) [30%] The solution of $x'' + x' = 0$, $x(0) = 1$, $x'(0) = 0$ is $x(t) = 1$. Show the details in Laplace's Method for obtaining this answer.

(b) [40%] Solve the system $x' = x - y$, $y' = y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method. Check the answer for y by the superposition shortcut for linear equations with constant coefficients.

(c) [30%] Find the Laplace transform of the convolution of $f(t) = e^t$ and $g(t) = t\cos t$.

Answer:

(a) The answer is $x(t) = 1$, by guessing, then checking the answer. Laplace details for Panel 1 end with equation $(s^2 + s)\mathcal{L}(x(t)) = s + 1$. Panel 2 begins with the division $\mathcal{L}(x(t)) = \dfrac{s+1}{s(s+1)} = \dfrac{1}{s}$. The backward table implies $\dfrac{1}{s} = \mathcal{L}(1)$. Lerch's cancelation theorem applies to give $x(t) = 1$.

(b) The transformed system is

$$\begin{array}{rcrcl} (s-1)\mathcal{L}(x) & + & (1)\mathcal{L}(y) & = & 0, \\ (0)\mathcal{L}(x) & + & (s-1)\mathcal{L}(y) & = & \mathcal{L}(2). \end{array}$$

Then forward table entry $\mathcal{L}(1) = 1/s$ and Cramer's Rule gives the formulas

$$\mathcal{L}(x) = \frac{-2}{s(s-1)^2)}, \quad \mathcal{L}(y) = \frac{2s-2}{s(s-1)^2}.$$

Use partial fractions:

$$\mathcal{L}(x) = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{(s-1)^2)}, \quad a = -2, b = 2, c = -2,$$

$$\mathcal{L}(y) = \frac{2}{s(s-1)} = \frac{a}{s} + \frac{b}{s-1}, \quad a = -2, b = 2.$$

The backward table implies

$$x = -2 + 2(1-t)e^t, \quad y = -2 + 2e^t.$$

Answer check. The superposition shortcut applied to $y' - y = 2$ gives $y = y_h + y + p$ where $y_h = c/e^{-t}$ and $y_p = -2$. Then $c = 2$ from $y(0) = 0$ and $y = -2 + 2e^t$.

(c) By the convolution theorem, the answer is the product of the Laplace transforms of f and g:

$$\text{Laplace of the convolution} = \mathcal{L}(e^t)\mathcal{L}(t\cos t)$$

Because the s-differentiation theorem applies to give $\mathcal{L}(t\cos t) = -\frac{d}{ds}\mathcal{L}(\cos t)$, then the answer is

$$\frac{1}{s-1}\frac{d}{ds}\frac{-s}{s^2+1} = \frac{s^2-1}{(s-1)(s^2+1)^2}.$$

Start your solution on this page.

5. **(Systems of Differential Equations)**

The eigenvalues of $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ are 3, 5 with corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) [20%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ by the eigenanalysis method. Please use symbols c_1, c_2 for the constants that appear in the general solution.

(b) [50%] Display the details for solution of $\mathbf{u}' = A\mathbf{u}$, $\vec{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, according to the Cayley-Hamilton-Ziebur shortcut.

The scalar form of the system is

$$\begin{cases} x'(t) &=& 4x(t) &+& y(t), \\ y'(t) &=& x(t) &+& 4y(t), \\ x(0) &=& 1, \\ y(0) &=& -1. \end{cases}$$

Please observe that the initial conditions evaluate constants, therefore the answer for (b) does not contain symbols c_1, c_2.

(c) [30%] A fundamental matrix $\Phi(t)$ for $\mathbf{u}' = A\mathbf{u}$ is a 2×2 invertible matrix such that $\Phi'(t) = A\Phi(t)$. Using the answer from either (a) or (b), find one fundamental matrix $\Phi(t)$ for the system $\mathbf{u}' = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \mathbf{u}$.

Answer:

(a) The eigenanalysis method implies the general solution is

$$\vec{u}(t) = c_1 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) The Cayley-Hamilton-Ziebur method says that in the scalar system the components $x(t), y(t)$ are linear combinations of the Euler solution atoms found from the roots of the characteristic equation $|C - rI| = 0$. The roots are $r = 3, 5$ and the atoms are e^{3t}, e^{5t}. Then

$$x(t) = c_1 e^{3t} + c_2 e^{5t}.$$

Solve the differential equation $x' = 4x + y$ for $y = x' - 4x$. Substitute the expression $x(t) = c_1 e^{3t} + c_2 e^{5t}$ to obtain the equation

$$y(t) = x' - 4x = 3c_1 e^{3t} + 5c_2 e^{5t} - 4(c_1 e^{3t} + c_2 e^{5t}) = -c_1 e^{3t} + c_2 e^{5t}.$$

Then the answer in terms of symbols c_1, c_2 is

$$x(t) = c_1 e^{3t} + c_2 e^{5t}, \quad y(t) = -c_1 e^{3t} + c_2 e^{5t}.$$

The equations that determine c_1, c_2 are $x(0) = 1, y(0) = -1$. Substitution into the solution above (set $t = 0$) gives the linear system of equations

$$1 = c_1 e^0 + c_2 e^0, \quad -1 = -c_1 e^0 + c_2 e^0.$$

Then $c_1 = 1, c_2 = 0$ and the answer is

$$x(t) = e^{3t}, \quad y(t) = -e^{3t}.$$

(c) The columns of the matrix Φ are independent solutions of the system $\mathbf{u}' = A\mathbf{u}$. These columns can be found from any general solution by differentiation on the symbols c_1, c_2. Therefore, by **(a)**, one possible fundamental matrix is

$$\Phi(t) = \begin{pmatrix} -e^{3t} & e^{5t} \\ e^{3t} & e^{5t} \end{pmatrix}.$$

Start your solution on this page.