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Differential Equations 2280

Sample Midterm Exam 1
Exam Date: Friday, 26 February 2016 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \dfrac{3+x^2}{1+x^2}$.

   (b) [25%] Solve $y' = (2\sin x + \cos x)(\sin x - 2\cos x)$.

   (c) [25%] Solve $y' = \dfrac{x\tan\left(\ln(1+x^2)\right)}{1+x^2}$, $y(0) = 2$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(t^2 v(t)) = 0$, $v(2) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = -20$.

   [Integral tables will be supplied for anything other than basic formulas. This sample problem would require no integral table. The exam problem will be shorter.]

(a) $y = \int \frac{3+x^2}{1+x^2}dx = \int \frac{2\,dx}{1+x^2} + \int 1\,dx = 2\tan^{-1}(x) + x + C$

(b) $y = \int (2\sin x + \cos x)(2\sin x + \cos x)'(-1)\,dx = -\frac{1}{2}(2\sin x + \cos x)^2 + C$

(c) $y = \int \frac{x\tan(\ln(1+x^2))}{1+x^2}dx$ $\qquad u = \ln(1+x^2)$, $du = \frac{2x}{1+x^2}dx$

$= \int \tan(u)\frac{du}{2}$

$= \frac{-1}{2}\ln(\cos(u)) + C$

$= -\frac{1}{2}\ln(\cos(\ln(1+x^2))) + C$

(d) $t^2 v(t) = C \Rightarrow 4v(2) = C \Rightarrow 40 = C$

$\boxed{v(t) = \frac{40}{t^2}}$

$x' = \frac{40}{t^2}$

$x = -40t^{-1} + C \Rightarrow -20 = -40/2 + C \Rightarrow C = 0$

$\boxed{x = -40/t}$

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form. No details expected.

$\square$	$y' + xy = y(2x + e^x) + x^2y$	$\square$	$y' = (x-1)(y+1) + (1-x)y$
$\square$	$y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$	$\square$	$y' + x^2e^y = xy$

(b) [10%] Is $y' + x(y+1) = ye^x + x$ separable? No details expected.

(c) [10%] Give an example of $y' = f(x, y)$ which is separable and linear but not quadrature. No details expected.

(d) [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

$\boxtimes$	$y' + xy = y(2x + e^x) + x^2y$	$\boxtimes$	$y' = (x-1)(y+1) + (1-x)y$
$\boxtimes$	$y' = 2e^{2x-y}e^{3y} + 3e^{3x+2y}$	$\square$	$y' + x^2e^y = xy$

(a) $y' + xy = 2xy + e^x y + x^2 y$ Linear, Separable
$y' = 2e^{2x}e^{2y} + 3e^{3x}e^{2y}$ Separable
$y' = xy - y + x - 1 + y - xy = x - 1$ SLQ
$y' = -x^2e^y + xy$ Not S, Q or L

(b) $y' = ye^x + x - xy - x = ye^x - xy = y(e^x - x)$
yes, Separable.

(c) $y' = xy$

(d) $f(x,y) = x + e^y$
$\frac{f_x}{f} = \frac{1}{x + e^y}$ not indep of y $\Rightarrow$ Not Separable
$f_y = e^y$ not indep of y $\Rightarrow$ Not linear

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3. (Solve a Separable Equation)
Given $(x+3)(y+1)y' = \Big((x+3)e^{-x+2} + 3x^2 + 2\Big)\,(y-1)(y+2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{y+1}{(y-1)(y+2)}\,y' = e^{2-x} + \frac{3x^2+2}{x+3}$$

$$\left(\frac{A}{y-1} + \frac{B}{y+2}\right) y' = e^{2-x} + 3x - 9 + \frac{29}{x+3}$$

integrate

$$\frac{2}{3}\ln|y-1| + \frac{1}{3}\ln|y+2| = -e^{-x+2} + \frac{3}{2}x^2 - 9x + 29\ln|x+3| + C$$

Long Division

$$\begin{array}{r|l} & 3x - 9 \\ \hline x+3 & 3x^2 + 2 \\ & 3x^2 + 9x \\ \hline & -9x + 2 \\ & -9x - 27 \\ \hline & 29 \end{array}$$

partial fractions

$y+1 = A(y+2) + B(y-1)$

$-1 = -3B$

$2 = 3A$

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4. (Linear Equations)

(a) [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{2t+3}x(t)$, $x(0) = 32$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x)y = 0$.

(c) [20%] Solve $5\frac{dy}{dx} + 10y = 7$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' + \frac{-5}{2t+3} = \frac{-160}{5}$, $\quad x(0) = 32$

$u = \int \frac{-5}{2t+3}\,dt$

$u = -\frac{5}{2}\ln|2t+3|$

$e^u = (2t+3)^{-5/2}$

$(e^u x)' = -32e^u$

$e^u x = -32\int (2t+3)^{-5/2}\,dt$

$= -32\,\frac{(2t+3)^{-3/2}}{(-3/2)(2)} + c$

$x = \frac{32}{3}(2t+3) + c(2t+3)^{5/2} \longrightarrow 32 = \frac{32}{3}(0+3) + c\,3^{5/2}$

$\longrightarrow c = 0$

$\boxed{x = \frac{64}{3}t + 32}$

(b) $y = \frac{c}{e^{-x^2}}$

(c) $y = \frac{7}{10} + \frac{c}{e^{2x}}$

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5} \left(|2x-1|-3\right)^3 (2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

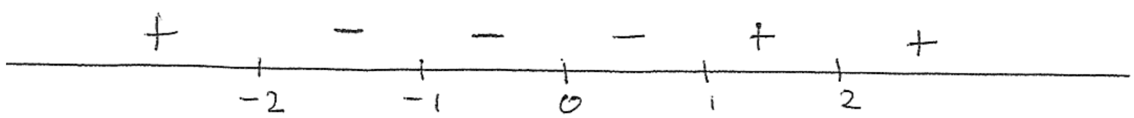
Expected in the phase line diagram are equilibrium points and signs of dx/dt.

Solution (a):

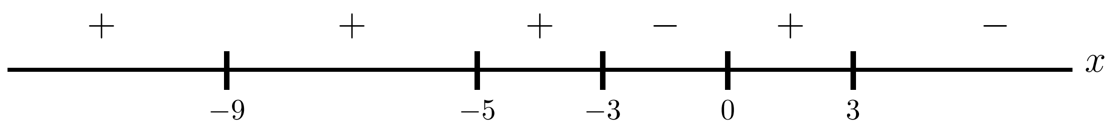
$$\frac{dx}{dt} = \left(\ln(1+5x^2)\right)^{1/5} \left(|2x-1|-3\right)^3 (2+x)^2(4-x^2)(1-x^2)^3 e^{\cos x}.$$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.

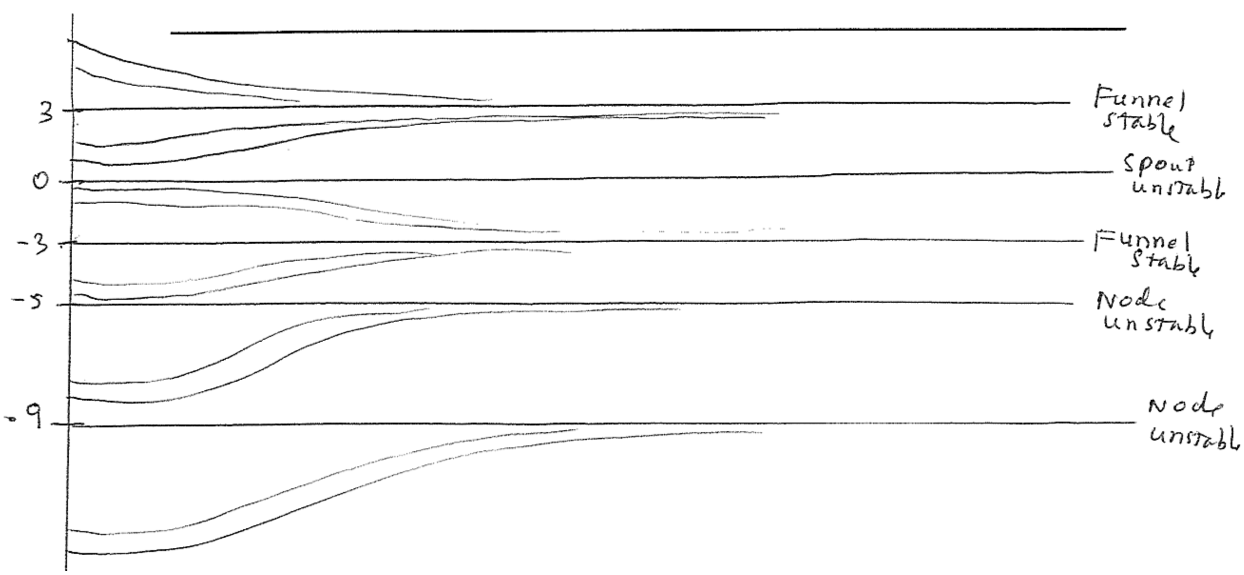
$x = 0$
$2x-1-3=0$
$2x-1+3=0$
$x+2=0$
$x-2=0$
$x=1$
$x=-1$



(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution (b):



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6. (ch3)
Using Euler's theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c) and (d).

(a) [25%] Find a differential equation $ay'' + by' + cy = 0$ with solutions $2e^{-x}$, $e^{-x} - e^{2x/3}$.

(b) [25%] Solve $y^{(6)} + 4y^{(5)} + 4y^{(4)} = 0$.

(c) [25%] Given characteristic equation $r(r+2)(r^3-4r)^3(r^2+2r+5) = 0$, solve the differential equation.

(d) [25%] Given $4x''(t) + 4x'(t) + 65x(t) = 0$, which represents an unforced damped spring-mass system with $m = 4$, $c = 4$, $k = 65$. Solve the differential equation [15%]. Classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a drawing of the physical model the meaning of constants m, c, k [5%].

Solution to Problem 6.

6(a)
Divide the first solution by 2. Then Euler atom e^{-x} is a solution, which implies that $r = -1$ is a root of the characteristic equation. Subtract $y_1 = e^{-x}$ and $y_2 = e^{-x} - e^{2x/3}$ to justify that $y = y_1 - y_2 = e^{2x/3}$ is a solution. It is an Euler atom corresponding to root $r = 2/3$. Then the characteristic equation should be $(r-(-1))(r-2/3) = 0$, or $3r^2+r-2 = 0$. The differential equation is $3y'' + y' - 2y = 0$.

6(b)
The characteristic equation factors into $r^4(r^2 + 4r + 4) = 0$ with roots $r = 0, 0, 0, 0, -2, -2$. Then y is a linear combination of the Euler atoms $1, x, x^2, x^3, e^{-2x}, xe^{-2x}$.

6(c)
The roots of the fully factored equation $r^4(r + 2)^4(r - 2)^3((r + 1)^2 + 4) = 0$ are

$$r = 0, 0, 0, 0, -2, -2, -2, -2, 2, 2, 2, -1 \pm 2i.$$

The solution y is a linear combination of the Euler atoms

$$1, x, x^2, x^3; \quad e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}; \quad e^{2x}, xe^{2x}, x^2e^{2x}; \quad e^{-x}\cos(2x), e^{-x}\sin(2x).$$

6(d)
Use $4r^2 + 4r + 65 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4i, -1/2 - 4i$. Case 2 of the recipe gives $y = (c_1 \cos 4t + c_2 \sin 4t)e^{-t/2}$. This is under-damped (it oscillates). The illustration shows a spring, dashpot and mass with labels k, c, m, x and the equilibrium position of the mass.

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7. (ch3)

(a) [25%] The trial solution y with fewest Euler solution atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(4)} + y^{(2)} = x + 2e^x + 3\sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients**! The trial solution should be the one with fewest Euler solution atoms.

Solution to Problem 7.

7(a). Rule I says the trial solution is $y = d_1 + d_2 x$. Rule II says to multiply by x until no atom is a solution of $y'' = 0$. Then $y = d_1 x^2 + d_2 x^3$ contains no terms of the homogeneous solution $y_h = c_1 + c_2 x$.

7(b). The homogeneous equation $y^{(4)} + y^{(2)} = 0$ has solution $y_h = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$, because the characteristic polynomial has roots 0, 0, i, $-i$.

$\boxed{\mathbf{1}}$ Rule I constructs an initial trial solution y from the list of independent Euler atoms

$$e^x, \quad 1, \quad x, \quad \cos x, \quad \sin x.$$

Linear combinations of these atoms are supposed to reproduce, by assignment of constants, all derivatives of $F(x) = x + 2e^x + 3\sin x$, which is the right side of the differential equation. Each of y_1 to y_4 in the display below is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of x. For example, $x = xe^{0x}$ strips to base atom e^{0x} or 1.

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 e^x, \\ y_2 &= d_2 + d_3 x, \\ y_3 &= d_4 \cos x, \\ y_4 &= d_5 \sin x. \end{aligned}$$

$\boxed{\mathbf{2}}$ Rule II is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 e^x, \\ y_2 &= x^2(d_2 + d_3 x), \\ y_3 &= x(d_4 \cos x), \\ y_4 &= x(d_5 \sin x). \end{aligned}$$

The powers of x multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. The factor used is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the related atom in the homogeneous solution y_h. The atom in y_1 is not a solution of the homogeneous equation, therefore y_1 is unaltered.