11.3 Structure of Linear Systems

Linear systems. A **linear system** is a system of differential equations of the form

$$
\begin{array}{rcccccccc}
x_1' & = & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & + & f_1, \\
x_2' & = & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & + & f_2, \\
\vdots & & & \vdots & \cdots & \vdots & & & \vdots \\
x_m' & = & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & + & f_m,
\end{array} \tag{1}
$$

where $' = d/dt$. Given are the functions $a_{ij}(t)$ and $f_j(t)$ on some interval $a < t < b$. The unknowns are the functions $x_1(t), \ldots, x_n(t)$.

The system is called **homogeneous** if all $f_j = 0$, otherwise it is called **non-homogeneous**.

Matrix Notation for Systems. A non-homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t),$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Existence-uniqueness. The fundamental theorem of Picard and Lindelöf applied to the matrix system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ says that a unique solution $\mathbf{x}(t)$ exists for each initial value problem and the solution exists on the common interval of continuity of the entries in $A(t)$ and $\mathbf{f}(t)$.

Three special results are isolated here, to illustrate how the Picard theory is applied to linear systems.

Theorem 3 (Unique Zero Solution)
Let $A(t)$ be an $m \times n$ matrix with entries continuous on $a < t < b$. Then the initial value problem

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{0}$$

has unique solution $\mathbf{x}(t) = \mathbf{0}$ on $a < t < b$.

Theorem 4 (Existence-Uniqueness for Constant Linear Systems)
Let $A(t) = A$ be an $m \times n$ matrix with constant entries and let $\mathbf{x}_0$ be any m-vector. Then the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

has a unique solution $\mathbf{x}(t)$ defined for all values of t.

Theorem 5 (Uniqueness and Solution Crossings)
Let $A(t)$ be an $m \times n$ matrix with entries continuous on $a < t < b$ and assume $\mathbf{f}(t)$ is also continuous on $a < t < b$. If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions of $\mathbf{u}' = A(t)\mathbf{u} + \mathbf{f}(t)$ on $a < t < b$ and $\mathbf{x}(t_0) = \mathbf{y}(t_0)$ for some t_0, $a < t_0 < b$, then $\mathbf{x}(t) = \mathbf{y}(t)$ for $a < t < b$.

Superposition. Linear homogeneous systems have **linear structure** and the solutions to nonhomogeneous systems obey a **principle of superposition**.

Theorem 6 (Linear Structure)
Let $\mathbf{x}' = A(t)\mathbf{x}$ have two solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$. If k_1, k_2 are constants, then $\mathbf{x}(t) = k_1\,\mathbf{x}_1(t) + k_2\,\mathbf{x}_2(t)$ is also a solution of $\mathbf{x}' = A(t)\mathbf{x}$.

The standard basis $\{\mathbf{w}_k\}_{k=1}^n$. The Picard-Lindelöf theorem applied to initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, with $\mathbf{x}_0$ successively set equal to the columns of the $n \times n$ identity matrix, produces n solutions $\mathbf{w}_1, \ldots, \mathbf{w}_n$ to the equation $\mathbf{x}' = A(t)\mathbf{x}$, all of which exist on the same interval $a < t < b$.

The linear structure theorem implies that for any choice of the constants $c_1, \ldots, c_n$, the vector linear combination

$$\mathbf{x}(t) = c_1\mathbf{w}_1(t) + c_2\mathbf{w}_2(t) + \cdots + c_n\mathbf{w}_n(t) \tag{2}$$

is a solution of $\mathbf{x}' = A(t)\mathbf{x}$.

Conversely, if $c_1, \ldots, c_n$ are taken to be the components of a given vector $\mathbf{x}_0$, then the above linear combination must be the unique solution of the initial value problem with $\mathbf{x}(t_0) = \mathbf{x}_0$. Therefore, all solutions of the equation $\mathbf{x}' = A(t)\mathbf{x}$ are given by the expression above, where $c_1, \ldots, c_n$ are taken to be **arbitrary constants**. In summary:

Theorem 7 (Basis)
The solution set of $\mathbf{x}' = A(t)\mathbf{x}$ is an n-dimensional subspace of the vector space of all vector-valued functions $\mathbf{x}(t)$. Every solution has a unique basis expansion (2).

Theorem 8 (Superposition Principle)
Let $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ have a particular solution $\mathbf{x}_p(t)$. If $\mathbf{x}(t)$ is any solution of $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$, then $\mathbf{x}(t)$ can be decomposed as **homogeneous plus particular**:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t).$$

The term $\mathbf{x}_h(t)$ is a certain solution of the homogeneous differential equation $\mathbf{x}' = A(t)\mathbf{x}$, which means arbitrary constants c_1, c_2, ... have been assigned certain values. The particular solution $\mathbf{x}_p(t)$ can be selected to be free of any unresolved or arbitrary constants.

Theorem 9 (Difference of Solutions)
Let $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ have two solutions $\mathbf{x} = \mathbf{u}(t)$ and $\mathbf{x} = \mathbf{v}(t)$. Define $\mathbf{y}(t) = \mathbf{u}(t) - \mathbf{v}(t)$. Then $\mathbf{y}(t)$ satisfies the homogeneous equation

$$\mathbf{y}' = A(t)\mathbf{y}.$$

General Solution. We explain **general solution** by example. If a formula $x = c_1 e^t + c_2 e^{2t}$ is called a general solution, then it means that all possible solutions of the differential equation are expressed by this formula. In particular, it means that a given solution can be represented by the formula, by specializing values for the constants c_1, c_2. We expect the number of arbitrary constants to be the least possible number.

The general solution of $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ is an expression involving arbitrary constants $c_1, c_2, \ldots$ and certain functions. The expression is often given in vector notation, although scalar expressions are commonplace and perfectly acceptable. Required is that the expression represents all solutions of the differential equation, in the following sense:

(a) Every **assignment of constants** produces a solution of the differential equation.

(b) Every possible solution is uniquely obtained from the expression by **specializing the constants**.

Due to the superposition principle, the constants in the general solution are identified as multipliers against solutions of the homogeneous differential equation. The general solution has some recognizable structure.

Theorem 10 (General Solution)
Let $A(t)$ be $n \times n$ and $\mathbf{f}(t)$ $n \times 1$, both continuous on an interval $a < t < b$. The linear nonhomogeneous system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ has general solution $\mathbf{x}$ given by the expression

$$\mathbf{x} = \mathbf{x}_h(t) + \mathbf{x}_p(t).$$

The term $\mathbf{y} = \mathbf{x}_h(t)$ is a general solution of the homogeneous equation $\mathbf{y}' = A(t)\mathbf{y}$, in which are to be found n arbitrary constants $c_1, \ldots, c_n$. The term $\mathbf{x} = \mathbf{x}_p(t)$ is a particular solution of $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$, in which there are present no unresolved nor arbitrary constants.

Recognition of homogeneous solution terms. An expression $\mathbf{x}$ for the general solution of a nonhomogeneous equation $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ involves arbitrary constants $c_1, \ldots, c_n$. It is possible to isolate both terms $\mathbf{x}_h$ and $\mathbf{x}_p$ by a simple procedure.

To find $\mathbf{x}_p$, set to zero all arbitrary constants $c_1, c_2, \ldots$; the resulting expression is free of unresolved and arbitrary constants.

To find $\mathbf{x}_h$, we find first the vector solutions $\mathbf{y} = \mathbf{u}_k(t)$ of $\mathbf{y}' = A(t)\mathbf{y}$, which are multiplied by constants c_k. Then the general solution $\mathbf{x}_h$ of the homogeneous equation $\mathbf{y}' = A(t)\mathbf{y}$ is given by

$$\mathbf{x}_h(t) = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) + \cdots + c_n\mathbf{u}_n(t).$$

Use partial derivatives on expression $\mathbf{x}$ to find the column vectors

$$\mathbf{u}_k(t) = \frac{\partial}{\partial c_k}\mathbf{x}.$$

This technique isolates the vector components of the homogeneous solution from any form of the general solution, including scalar formulas for the components of $\mathbf{x}$. In any case, the general solution $\mathbf{x}$ of the linear system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ is represented by the expression

$$\mathbf{x} = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) + \cdots + c_n\mathbf{u}_n(t) + \mathbf{x}_p(t).$$

In this expression, each *assignment* of the constants $c_1, \ldots, c_n$ produces a solution of the nonhomogeneous system, and conversely, each possible solution of the nonhomogeneous system is obtained by a unique *specialization* of the constants $c_1, \ldots, c_n$.

To illustrate the ideas, consider a 3×3 linear system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ with general solution

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

given in scalar form by the expressions

$$\begin{aligned} x_1 &= c_1e^t + c_2e^{-t} + t, \\ x_2 &= (c_1 + c_2)e^t + c_3e^{2t}, \\ x_3 &= (2c_2 - c_1)e^{-t} + (4c_1 - 2c_3)e^{2t} + 2t. \end{aligned}$$

To find the vector form of the general solution, we take partial derivatives $\mathbf{u}_k = \dfrac{\partial \mathbf{x}}{\partial c_k}$ with respect to the variable names c_1, c_2, c_3:

$$\mathbf{u}_1 = \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix}.$$

To find $\mathbf{x}_p(t)$, set $c_1 = c_2 = c_3 = 0$:

$$\mathbf{x}_p(t) = \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}.$$

Finally,

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) + c_3\mathbf{u}_3(t) + \mathbf{x}_p(t) \\ &= c_1 \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}.\end{aligned}$$

The expression $\mathbf{x} = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) + c_3\mathbf{u}_3(t) + \mathbf{x}_p(t)$ satisfies required elements (a) and (b) in the definition of general solution. We will develop now a way to routinely test the uniqueness requirement in (b).

Independence. Constants $c_1, \ldots, c_n$ in the general solution $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$ appear exactly in the expression $\mathbf{x}_h$, which has the form

$$\mathbf{x}_h = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

A solution $\mathbf{x}$ uniquely determines the constants. In particular, the zero solution of the homogeneous equation is uniquely represented, which can be stated this way:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0} \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

This statement is equivalent to the statement that the vector-valued functions $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ are **linearly independent**.

It is possible to write down a candidate general solution to some 3×3 linear system $\mathbf{x}' = A\mathbf{x}$ via equations like

$$\begin{aligned} x_1 &= c_1e^t + c_2e^t + c_3e^{2t}, \\ x_2 &= c_1e^t + c_2e^t + 2c_3e^{2t}, \\ x_3 &= c_1e^t + c_2e^t + 4c_3e^{2t}. \end{aligned}$$

This example was constructed to contain a classic mistake, for purposes of illustration.

How can we detect a mistake, given only that this expression is supposed to represent the general solution? First of all, we can test that $\mathbf{u}_1 = \partial\mathbf{x}/\partial c_1$, $\mathbf{u}_2 = \partial\mathbf{x}/\partial c_2$, $\mathbf{u}_3 = \partial\mathbf{x}/\partial c_3$ are indeed solutions. But to insure the unique representation requirement, the vector functions $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ must be linearly independent. We compute

$$\mathbf{u}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}.$$

Therefore, $\mathbf{u}_1 = \mathbf{u}_2$, which implies that the functions $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ fail to be independent. While is is possible to test independence by a rudimentary test based upon the definition, we prefer the following test due to Abel.

Theorem 11 (Abel's Formula and the Wronskian)
Let $\mathbf{x}_h(t) = c_1\mathbf{u}_1(t) + \cdots + c_n\mathbf{u}_n(t)$ be a candidate general solution to the equation $\mathbf{x}' = A(t)\mathbf{x}$. In particular, the vector functions $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ are solutions of $\mathbf{x}' = A(t)\mathbf{x}$. Define the **Wronskian** by

$$w(t) = \det(\mathbf{aug}(\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t))).$$

Then **Abel's formula** holds:

$$w(t) = e^{\int_{t_0}^{t} \mathbf{trace}(A(s))ds} w(t_0).^5$$

In particular, $w(t)$ is either everywhere nonzero or everywhere zero, accordingly as $w(t_0) \neq 0$ or $w(t_0) = 0$.

Theorem 12 (Abel's Wronskian Test for Independence)
The vector solutions $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\mathbf{x}' = A(t)\mathbf{x}$ are independent if and only if the Wronskian $w(t)$ is nonzero for some $t = t_0$.

Clever use of the point t_0 in Abel's Wronskian test can lead to succinct independence tests. For instance, let

$$\mathbf{u}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}.$$

Then $w(t)$ might appear to be complicated, but $w(0)$ is obviously zero because it has two duplicate columns. Therefore, Abel's Wronskian test detects **dependence** of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

To illustrate Abel's Wronskian test when it detects independence, consider the column vectors

$$\mathbf{u}_1 = \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix}.$$

At $t = t_0 = 0$, they become the column vectors

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Then $w(0) = \det(\mathbf{aug}(\mathbf{u}_1(0), \mathbf{u}_2(0), \mathbf{u}_3(0))) = 1$ is nonzero, testing **independence** of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

[5] The **trace** of a square matrix is the sum of its diagonal elements. In literature, the formula is called the **Abel-Liouville** formula.

Initial value problems and the rref method. An **initial value problem** is the problem of solving for $\mathbf{x}$, given

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

If a general solution is known,

$$\mathbf{x} = c_1\mathbf{u}_1(t) + \cdots + c_n\mathbf{u}_n(t) + \mathbf{x}_p(t),$$

then the problem of finding $\mathbf{x}$ reduces to finding $c_1, \ldots, c_n$ in the relation

$$c_1\mathbf{u}_1(t_0) + \cdots + c_n\mathbf{u}_n(t_0) + \mathbf{x}_p(t_0) = \mathbf{x}_0.$$

This is a matrix equation for the unknown constants $c_1, \ldots, c_n$ of the form $B\mathbf{c} = \mathbf{d}$, where

$$B = \mathbf{aug}(\mathbf{u}_1(t_0), \ldots, \mathbf{u}_n(t_0)), \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \mathbf{d} = \mathbf{x}_0 - \mathbf{x}_p(t_0).$$

The **rref**-method applies to find $\mathbf{c}$. The method is to perform swap, combination and multiply operations to $C = \mathbf{aug}(B, \mathbf{d})$ until $\mathbf{rref}(C) = \mathbf{aug}(I, \mathbf{c})$.

To illustrate the method, consider the general solution

$$\begin{aligned} x_1 &= c_1e^t + c_2e^{-t} + t, \\ x_2 &= (c_1 + c_2)e^t + c_3e^{2t}, \\ x_3 &= (2c_2 - c_1)e^{-t} + (4c_1 - 2c_3)e^{2t} + 2t. \end{aligned}$$

We shall solve for c_1, c_2, c_3, given the initial condition $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = -1$. The above relations evaluated at $t = 0$ give the system

$$\begin{aligned} 1 &= c_1e^0 + c_2e^0 + 0, \\ 0 &= (c_1 + c_2)e^0 + c_3e^0, \\ -1 &= (2c_2 - c_1)e^0 + (4c_1 - 2c_3)e^0 + 0. \end{aligned}$$

In standard scalar form, this is the 3×3 linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & & & = & 1, \\ c_1 & + & c_2 & + & c_3 & = & 0, \\ 3c_1 & + & 2c_2 & - & 2c_3 & = & -1. \end{array}$$

The augmented matrix C, to be reduced to **rref** form, is given by

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & -1 \end{pmatrix}.$$

After the **rref** process is completed, we obtain

$$\mathbf{rref}(C) = \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

From this display, we read off the answer $c_1 = -5$, $c_2 = 6$, $c_3 = -1$ and report the final answer

$$\begin{aligned} x_1 &= -5e^t + 6e^{-t} + t, \\ x_2 &= e^t - e^{2t}, \\ x_3 &= 17e^{-t} - 18e^{2t} + 2t. \end{aligned}$$

Equilibria. An equilibrium point $\mathbf{x}_0$ of a linear system $\mathbf{x}' = A(t)\mathbf{x}$ is a constant solution, $\mathbf{x}(t) = \mathbf{x}_0$ for all t. Mostly, this makes sense when $A(t)$ is constant, although the definition applies to continuous systems. For a solution $\mathbf{x}$ to be constant means $\mathbf{x}' = \mathbf{0}$, hence all equilibria are determined from the equation

$$A(t)\mathbf{x}_0 = \mathbf{0} \quad \text{for all } t.$$

This is a homogeneous system of linear algebraic equations to be solved for $\mathbf{x}_0$. It is not allowed for the answer $\mathbf{x}_0$ to depend on t (if it does, then it is **not** an equilibrium). The theory for a constant matrix $A(t) \equiv A$ says that either $\mathbf{x}_0 = \mathbf{0}$ is the unique solution or else there are infinitely many answers for $\mathbf{x}_0$ (the nullity of A is positive).