Atom List. L. Euler supplies us with a basic result, which tells us how to find the list of distinct atoms.

Theorem 1 (Euler)
The function e^{rx} is a solution of a linear constant coefficient differential equation if and only if r is a root of the characteristic equation.
More generally, the list of distinct atoms e^{rx}, xe^{rx}, $\ldots$, x^ke^{rx} consists of solutions if and only if r is a root of the characteristic equation of multiplicity $k+1$.
If $r = \alpha + i\beta$ is a complex root of multiplicity $k+1$, then the formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$e^{rx} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x).$$

Therefore, the $2k+2$ distinct atoms listed below are independent solutions of the differential equation:

$$\begin{array}{l} e^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\cos(\beta x),\ \ldots,\ x^ke^{\alpha x}\cos(\beta x), \\ e^{\alpha x}\sin(\beta x),\ xe^{\alpha x}\sin(\beta x),\ \ldots,\ x^ke^{\alpha x}\sin(\beta x) \end{array}$$

1 Example (First Order) Solve $2y' + 5y = 0$ by using the nth order recipe, showing $y_h = c_1 e^{-5x/2}$.

Solution: The characteristic equation is $2r + 5 = 0$ with real root $r = -5/2$ and corresponding atom e^{rx} given explicitly by $e^{-5x/2}$. Euler's Theorem was applied here. The order of the differential equation is 1, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$, and therefore $y_h = c_1 e^{-5x/2}$.

2 Example (Second Order I) Solve $y'' + 2y' + y = 0$ by using the nth order recipe, showing $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

Solution: The characteristic equation is $r^2 + 2r + 1 = 0$ with double real root $r = -1, -1$. Euler's Theorem applies to report atom list e^{rx}, xe^{rx}, given explicitly by e^{-x}, xe^{-x}. The order of the differential equation is 2, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$, and therefore $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

3 Example (Second Order II) Solve $y'' + 3y' + 2y = 0$ by using the nth order recipe, showing $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

Solution: The characteristic equation is $r^2 + 3r + 2 = 0$ with distinct real roots $r_1 = -1$, $r_2 = -2$. Euler's Theorem applies to report atom list $e^{r_1 x}$, $e^{r_2 x}$, given explicitly by e^{-x}, e^{-2x}. The order of the differential equation is 2, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$, and therefore $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

4 Example (Second Order III) Solve $y'' + 2y' + 5y = 0$ by using the nth order recipe, showing $y_h = c_1 e^{-x}\cos 2x + c_2 x e^{-x}\sin 2x$.

Solution: The characteristic equation is $r^2 + 2r + 5 = 0$ with complex conjugate roots $r_1 = -1 + 2i$, $r_2 = -1 - 2i$. Euler's Theorem applies to report an atom list $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$, where $\alpha = -1$, $\beta = 2$ are the real and imaginary parts of the root $\alpha + i\beta = -1 + 2i$ (then $\alpha = -1$, $\beta = 2$). The atom list is given explicitly by $e^{-x}\cos 2x$, $e^{-x}\sin 2x$. The order of the differential equation is 2, so we have found all atoms. The lesson: applying Euler's theorem to the second conjugate root $-1 - 2i$ will produce no new atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x$.

5 Example (Third Order I) Solve $y''' - y' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 e^x + c_3 e^{-x}$.

Solution: The characteristic equation is $r^3 - r = 0$ with real roots $r_1 = 0, r_2 = 1, r_3 = -1$. Euler's Theorem applies to report atom list $e^{r_1 x}$, $e^{r_2 x}$, $e^{r_3 x}$ given explicitly by e^{0x}, e^x, e^{-x}. The order of the differential equation is 3, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants $c_1, c_2, \ldots$, and therefore $y_h = c_1 e^{0x} + c_2 e^x + c_3 e^{-x}$. Convention dictates replacing e^{0x} by 1 in the final equation.

6 Example (Third Order II) Solve $y''' - y'' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 x + c_3 e^x$.

Solution: The characteristic equation is $r^3 - r^2 = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. Euler's Theorem applies to report atom list $e^{r_1 x}$, $xe^{r_1 x}$, $e^{r_3 x}$ given explicitly by e^{0x}, xe^{0x}, e^x. The order of the differential equation is 3, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, $\ldots$, and therefore $y_h = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^x$. Convention dictates replacing e^{0x} by 1 in the final equation.

7 Example (Fourth Order) Solve $y^{iv} - y'' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Solution: The characteristic equation is $r^4 - r^2 = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$, $r_4 = -1$. Euler's Theorem applies to obtain the atom list $e^{r_1 x}$, $xe^{r_1 x}$, $e^{r_3 x}$, $e^{r_4 x}$, given explicitly by e^{0x}, xe^{0x}, e^x, e^{-x}. The order of the differential equation is 4, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, $\ldots$, and therefore $y_h = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^x + c_4 e^{-x}$. Convention replaces e^{0x} by 1 in the final equation.