Linear Algebra and Music Theory

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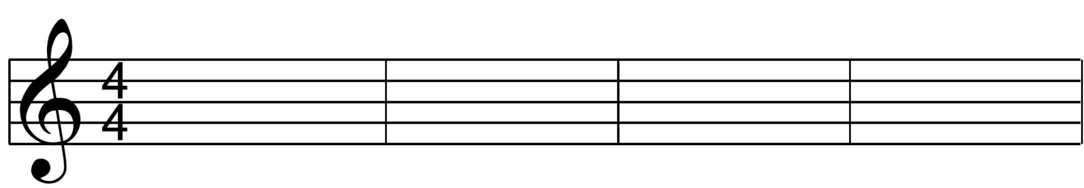
The relation to music and Mathematics is quite a thick topic to discuss because, in a very general sense, music is mathematics. It is a difference and consistency of changes in melody and rhythm that generates sound in a way that we enjoy and appreciate it. Many different applications have been used to describe these changes from logarithmic fluctuation in sound wave frequency to the difference and scale relations of transposed music. Besides what relations we know of linear algebra and music already, I wish to take an unconventional look at the use of Linear Algebra in music composition by representing music in the form of a matrix instead of on a staff and clef.

This topic interests me in particular because I have always loved composing music and the differences in melodies via Music theory. I wasn't sure how to represent it until I saw a video online in which a musician used a physical programmable matrix to represent his music. It turns out that such a programmable model such as that is very common among many musical instruments such as music boxes. Through this, and after more studying and some of my own experiments, derived various ideas on how Linear Algebra can be used to represent and manipulate music on a theoretical level.

Representing Music in a Matrix

Normally music is represented by a staff and clef as in what follows:



The 5 horizontal bars, or staff, represent the progressive steps between notes. The treble clef signifies how the staffs represent the notes (or frequencies) while the two 4's represent the time signature or how the rhythm is divided (also a tempo is used to represent time given in beats per minute). The bars break up the amount of notes that can be played within each section with the time signature in mind. From there, notes (♩) are place in sequential order to determine the music with a change in structure (♪) to represent the difference in melody and length.

With variations on the use of each of these elements, this is the traditional way of writing and reading music. In a sense, it is a matrix already only transposed from our usual view. To display this in the format of a matrix, I will be representing by a 1 x 12 matrix as follows:

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

The matrix rows can represent the number of beats per-section similar to the 4 by 4 time signature, this will be come more important further on. The 12 columns represent the 12 notes or frequencies in a scale that are detectable by human ears to the octave. The octave (not included) is a note that is exactly half the frequency of the starting, that is significant because they are the same note, yet at a different pitch. Melody and notes will be used as numbers on the matrix. A 1 will represent as a quarter note per row and can be increased in number to signify volume and intensity. Just as how the staff and signatures change on the traditional representation of music, our matrix can be adjusted to fit our needs such as increasing the columns to represent more notes and changing the rows to more time or timing representation.

Now with our definition of a matrix set, we can begin exploring the properties of how music theory is represented in our matrix.

Musical Theory and Matrix Rank

Music theory generally revolves around a set of chords that progress through each note of the music scale. With the scale here in matrix form being the reference I'd like to discuss the different set of chords and how they are perceived in matrix notation:

Major Scale:	[1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1]
I :	[1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0]
II :	[0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0]
III :	[0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1]

IV : [1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0]

V : [0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0]

VI : [1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0]

VII: [0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1]

When I – VII are added together, the resulting matrix. Is as follows:

[3, 0, 3, 0, 3, 3, 0, 3, 0, 3, 0, 3]

Which in reality, is the original major matrix multiplied by 3, simple enough. What is interesting is that if each of these each were represented as vectors, we would only need three of them to span R^7, the same rank as the original Major scale.

What makes this significant, is that most modern songs, in simplicity, are based off of a 3 or 4 chord system which together usually represent R^7. For some reason this has a musical appeal to us as a human race that we play songs that, if chords are put together right, represent the same dimension and span as the key the song is played in.

To demonstrate this, here are 4 chords that are *very* commonly used in many songs. The most familiar being *Lollipop* by *Chordettes*:

I : [1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0]

VI : [1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0]

IV : [1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0]

V : [0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0]

Sure enough added together, these create the major scale with a few fluctuations:

[3, 0, 1, 0, 2, 1, 0, 1, 0, 2, 0, 1]

This isn't to say that we always universally enjoy this span and combination of music, there are plenty of counter examples. However it is such a reoccurring element in our music that it is important to notice. Some of the greatest considered music on earth follows this course in the many compositions of the mathematical Musical Genius, Johann Sebastian Bach. His many Compositions are still revered today as calculative masterpieces and in particular, one of my favorites, his *Cello Prelude in C major* represents this principle beautifully. The song not only plays such chords to represent the key scale, but also if there is an irregularity (for example, a lack of use of the 6th variable) the song makes adjustments to existing chords (a change from a G, to a G^7) to meet its demand and balance it to be in closer relation with the other existing variables.

So, to summarize simply, we enjoy music progressions with chords that share the same rank as their key scale.

Jazz Complexity and Linear Independence

Some of my favorite pieces of music are of Jazz improvisation. The number of abnormal complexities in the style make it possible to continue playing for as long as the improviser desires. This is because of the range and scaling that Jazz progressions cover. In similarity to the previous point, appealing Jazz progressions usually cover all the notes of a scale, however they go a step further and cover all variables of the matrix. So to say, usually in a set of Jazz progressions, their corresponding matrices usually result in Linear Independence of the Matrix.

Here is a favorite progression (the key of A) to explain:

EbM7: [0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0]
Bb7: [0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0]
AbM7: [1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0]
Ab7: [1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0]
Db7: [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1]
C7: [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0]

Added together they form a solid matrix without missing variables:

[3, 1, 2, 3, 1, 2, 1, 3, 4, 1, 2, 1];

Even more, most jazz progressions are played in 6 chords in a usual 2 part succession make it a 12 chord progression total. At times the second part will be a different set of chords, but together they make a 12 by 12 matrix that is typically linearly independent. Though this may not be true for all Jazz progressions, the typical of them follow this type of pattern.

What We Find Boring

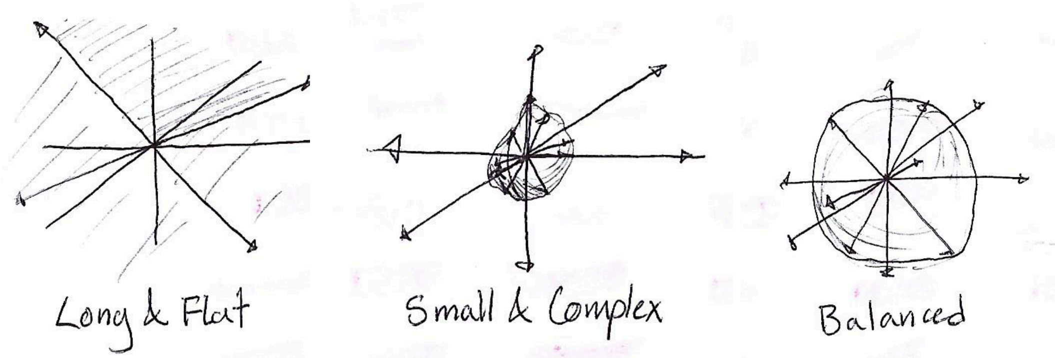
Now with such an idea of knowing that the higher the rank of a piece of music makes it more appealing to the human ear, what about additional properties of this musical matrix? What other components and combinations of music do we find appealing? To answer such questions, it would actually help to search what would be considered the contrary: boring or even annoying.

It is understandable that music is an art and it appeals to people differently based upon their tastes and interests. In addition to this, there are many songs that 'break the rules' and yet are still appealing. However, the focus of this is on the aggregate of our appreciation of music. I attempt to ask, what are these rules we are following or breaking? It is important to know the rules of your subject in order to surpass them and yet still succeed. So what happens when you break the rules and fail?

A very common occurrence of 'bad' music is a high repetition of notes and chords. We find music 'repetitive' and 'predictable' making it unpleasant to the human ear. This can even from the perspective of each chord representing a vector. Combined together, these chord vectors of repetitive music in a Cartesian coordinate system will seem "flat" or "shallow" compared to it's possible complexity. They will have great lengths yet represent very little of the musical spectrum.

On the contrary, music that is too complex and contains too many directions in such a small amount of vector will begin to confuse and even irritate the listener. Of course in this model here is a higher rank in the musical representation, however the amount of vectors need to do so only generates a small 'area' of music.

Imagine for a minute that our music model is fully represented to the third degree instead of the twelfth. Now imagine a song that consisted of only one note and that played consistently, it would generate a very long line from the origin. Now if there was a song that generated only 3 vectors, one with many values and pointed completely orthogonal to each other. They would hold a high rank, but their limits would be very confined to a small area.



Example of how vectors of composition are represented

A both long and complex piece of music would generate a large and deep area and their limits would be large and spacious. This is generally the characteristic of good musical composition, if it covers a larger area yet is still complex in is chord progression. Where this could lead though, would be the development of a vector space that define the “boundaries” of good music. Much more research and development would be needed to solidify this concept, but it would be intriguing to see what would be defined in a mathematical sense what “good music” is.

Conclusion

There are many other routes to be taken with Music and it’s relationship to Linear Algebra, but in terms of musical composition, it seems there isn’t much research. Perhaps it’s due to the lack of coordination to the many rules of Linear Algebra, but I’m sure that through study, it could result in some surprising claims. I think that my claims so far have some value, but there is always more research to be done.

So in summary, if musical composition is represented in vectors, it has some common distinct relationship to how we appreciate it. The first being, the rank must remain the same as the given “key signature”, second the higher the rank in a given scale provides a greater appreciation for the music, and third there are given “boundaries” and a vector space in which music becomes appealing to the human ear. Given these attributes and with just a hint of your own creativity, anyone could compose music that could be appealing and even inspiring.