

## Linear Algebra 2270-2

Due in Week 9

The ninth week finishes chapter 4 and starts the work from chapter 5. Here's the list of problems, problem notes and answers.

**Section 4.4.** Exercises 3, 7, 9, 13, 27

**Section 4.5.** Exercises 5, 7, 11, 13, 21

**Section 4.6.** Exercises 1, 3, 5, 7, 15, 21

**Extra Credit Problem week9-1.** Define a function  $T$  from  $\mathcal{R}^n$  to  $\mathcal{R}^m$  by the matrix multiply formula  $T(\vec{x}) = A\vec{x}$ . Prove that for all vectors  $\vec{u}, \vec{v}$  and all constants  $c$ , (a)  $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ , (b)  $T(c\vec{u}) = cT(\vec{u})$ . *Definition:*  $T$  is called a **linear transformation** if  $T$  maps  $\mathcal{R}^n$  into  $\mathcal{R}^m$  and satisfies (a) and (b).

**Extra Credit Problem week9-2.** Let  $T$  be a linear transformation from  $\mathcal{R}^n$  into  $\mathcal{R}^n$  that satisfies  $\|T(\vec{x})\| = \|\vec{x}\|$  for all  $\vec{x}$ . Prove that the  $n \times n$  matrix  $A$  of  $T$  is orthogonal, that is,  $A^T A = I$ , which means the columns of  $A$  are **orthonormal**:

$$\text{col}(A, i) \cdot \text{col}(A, j) = 0 \quad \text{for } i \neq j, \quad \text{and} \quad \text{col}(A, i) \cdot \text{col}(A, i) = 1.$$

**Extra Credit Problem week9-3.** Let  $T$  be a linear transformation given by  $n \times n$  orthogonal matrix  $A$ . Then  $\|T(\vec{x})\| = \|\vec{x}\|$  holds. Construct an example of such a matrix  $A$  for dimension  $n = 3$ , which corresponds to holding the  $z$ -axis fixed and rotating the  $xy$ -plane 45 degrees counter-clockwise. Draw a 3D-figure which shows the action of  $T$  on the unit cube  $S = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ .

### Problem Notes

**Extra Credit Problem week9-1.** Write out both sides of identities (a) and (b), replacing  $T(\vec{w})$  by matrix product  $A\vec{w}$  for various choices of  $\vec{w}$ . Then compare sides to finish the proof.

**Extra Credit Problem week9-2.** Equation  $\|T(\vec{x})\| = \|\vec{x}\|$  means lengths are preserved by  $T$ . It also means  $\|A\vec{x}\| = \|\vec{x}\|$ , which applied to  $\vec{x} = \text{col}(I, k)$  means  $\text{col}(A, k)$  has length equal to  $\text{col}(I, k)$  (=1). Write  $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w} = \vec{w}^T \vec{w}$  (the latter a matrix product). Then write out the equation  $\|A\vec{x}\|^2 = \|\vec{x}\|^2$ , to see what you get, for various choices of unit vectors  $\vec{x}$ .

**Extra Credit Problem week9-3.** The equations for such a transformation can be written as plane rotation equations in  $x, y$  plus the identity in  $z$ . They might look like  $x' = x \cos \theta - y \sin \theta$ ,  $y' =$  similar,  $z' = z$ . Choose  $\theta$  then test it by seeing what happens to  $x = 1, y = 0, z = 0$ , the answer for which is a rotation of vector  $(1, 0, 0)$ . The answer for  $A$  is obtained by writing the scalar equations as a matrix equation  $(x', y', z')^T = A(x, y, z)^T$ .