MATH 2270-2 Final Exam Sample Problems Spring 2016

1. (5 points) Let A be a 2×2 matrix such that $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Compute $A \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

2. (5 points) State (1) the definition of norm, (2) the Cauchy-Schwartz inequality and (3) the triangle inequality, for vectors in $\mathcal{R}^n$.

3. (5 points) Suppose $A = B(C + D)E$ and all the matrices are $n \times n$ invertible. Find an equation for C.

4. (5 points) Find all solutions to the system of equations

$$\begin{aligned} 2w + 3x + 4y + 5z &= 1 \\ 4w + 3x + 8y + 5z &= 2 \\ 6w + 3x + 8y + 5z &= 1 \end{aligned}$$

5. (5 points) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Show the details of two different methods for finding the inverse of the matrix A.

6. (5 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

7. (5 points) Let Q be a 2×2 matrix with $QQ^T = I$. Prove that Q has columns of unit length and its two columns are orthogonal.

8. (5 points) True or False? If the 3×3 matrices A and B are triangular, then AB is triangular.

9. (5 points) True or False? If a 3×3 matrix A has an inverse, then for all vectors $\vec{b}$ the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.

10. (5 points) Let A be a 3×4 matrix. Find the elimination matrix E which under left multiplication against A performs both (1) and (2) with one matrix multiply.

(1) Replace Row 2 of A with Row 2 minus Row 1.

(2) Replace Row 3 of A by Row 3 minus 5 times Row 2.

11. (10 points) Determinant problem, chapter 3.
(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.
(b) [10%] True or False? The determinant of the negative of the $n \times n$ identity matrix is -1.
(c) [30%] Assume given 3×3 matrices A, B. Suppose $E_2 E_1 A^2 = AB$ and E_1, E_2 are elementary matrices representing respectively a combination and a multiply by 3. Assume $\det(B) = 27$. Let $C = -A$. Find all possible values of $\det(C)$.
(d) [20%] Determine all values of x for which $(2I + C)^{-1}$ fails to exist, where I is the 3×3 identity and $C = \begin{pmatrix} 2 & x & -1 \\ 3x & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.
(e) [30%] Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & \frac{1}{2} \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 4, column 2 of A^{-1}.

12. (5 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 0 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_3 by Cramer's Rule, showing **all details** (details count 75%). To save time, **do not compute** x_1, x_2!

13. (5 points) Define matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

14. (5 points) Determine which values of k correspond to a unique solution for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

15. (10 points) Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & -b & c \\ 1 & c & a \\ 2 & -b+c & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(b + c)(2a + c) \neq 0$.

(b). The system has no solution if $2a+c = 0$ and $a \neq 0$ (don't explain the other possibilities).

(c). The system has infinitely many solutions if $a = c = 0$ (don't explain the other possibilities).

16. (5 points) Explain how the **span theorem** applies to show that the set S of all linear combinations of the functions $\cosh x, \sinh x$ is a subspace of the vector space V of all continuous functions on $-\infty < x < \infty$.

17. (5 points) Write a proof that the subset S of all solutions $\vec{x}$ in $\mathcal{R}^n$ to a homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathcal{R}^n$. This is called the **kernel theorem**.

18. (5 points) Using the subspace criterion, write two hypotheses that imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

19. (5 points) Report which columns of A are pivot columns: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

20. (5 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

21. (5 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

22. (5 points) Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

23. (5 points) A 10×13 matrix A is given and the homogeneous system $A\vec{x} = \vec{0}$ is transformed to reduced row echelon form. There are 7 lead variables. How many free variables?

24. (5 points) The rank of a 10×13 matrix A is 7. Find the nullity of A.

25. (5 points) Given a basis $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ of $\mathcal{R}^2$, and $\vec{v} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for a unique set of coefficients c_1, c_2, called the *coordinates of* $\vec{v}$ *relative to the basis* $\vec{v}_1, \vec{v}_2$. Compute c_1 and c_2.

26. (5 points) Determine independence or dependence for the list of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

27. (5 points) Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Euler Atom test**	Any finite set of distinct atoms is independent.
☐	**Sample test**	Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Orthogonality test**	A set of nonzero pairwise orthogonal vectors is independent.

28. (5 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

29. (5 points) The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

30. (5 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthonormal basis of S.

31. (5 points) Find the orthogonal projection vector $\vec{v}$ (the shadow projection vector) of

$\vec{v}_2$ onto $\vec{v}_1$, given

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

32. (5 points) Let A be an $m \times n$ matrix with independent columns. Prove that $A^T A$ is invertible.

33. (5 points) Let A be an $m \times n$ matrix with $A^T A$ invertible. Prove that the columns of A are independent.

34. (5 points) Let A be an $m \times n$ matrix and $\vec{v}$ a vector orthogonal to the nullspace of A. Prove that $\vec{v}$ must be in the row space of A.

35. (5 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

36. (5 points) Assume $A^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the inverse of the transpose of A.

37. (5 points) This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$..

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $C^{-1}B^2$.

$$B = \begin{pmatrix} 1 & -1 & ? & ? \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 2 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

38. (5 points) Display the entry in row 3, column 4 of the adjugate matrix [or adjoint matrix] of $A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Report both the symbolic formula and the numerical value.

39. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

40. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

41. (5 points) The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$.
To save time, **don't find the eigenvector for** $\lambda = 0$.

42. (5 points) Find the two complex eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

43. (5 points) Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

44. (5 points) Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

45. (5 points) Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

46. (5 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display three eigenpairs of A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

47. (5 points) Assume A is a given 4×4 matrix with eigenvalues 0, 1, $3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

48. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

49. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

50. (5 points) Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

51. (5 points) Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

52. (5 points) Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

53. (5 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $\hat{\vec{\mathbf{b}}}$ be the near point to $\vec{\mathbf{b}}$ in the subspace W. Find $\hat{\vec{\mathbf{b}}}$.

54. (5 points) There are real 2×2 matrices A such that $A^2 = -4I$, where I is the identity matrix. Give an example of one such matrix A and then verify that $A^2 + 4I = 0$.

55. (5 points) Let $Q = < \vec{q}_1 | \vec{q}_2 >$ be orthogonal 2×2 and D a diagonal matrix with diagonal entries λ_1, λ_2. Prove that the 2×2 matrix $A = QDQ^T$ satisfies $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T$.

56. (5 points) A matrix A is **defined** to be positive definite if and only if $\vec{x}^T A \vec{x} > 0$ for nonzero $\vec{x}$. Which of these matrices are positive definite?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix}$$

57. (5 points) Let A be a real symmetric 2×2 matrix. Prove that the eigenvalues of A are real numbers.

58. (5 points) Let B be a real 3×4 matrix. Prove that the eigenvalues of $B^T B$ are non-negative.

59. (5 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$.

60. (5 points) Show that if B is an invertible matrix and A is similar to B, with $A = PBP^{-1}$, then A is invertible.

61. (5 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

62. (5 points) Strang's Four Fundamental Subspaces are the nullspace of A, the nullspace of A^T, the row space of A and the column space of A. Describe, using a figure or drawing, the locations in the matrices U, V of the singular value decomposition $A = U\Sigma V^T$ which are consumed by the four fundamental subspaces of A.

63. (5 points) Give examples for a vertical shear and a horizontal shear in the plane. Expected is a 2×2 matrix A which represents the linear transformation.

64. (5 points) Give examples for clockwise and counterclockwise rotations in the plane. Expected is a 2×2 matrix A which represents the linear transformation.

65. (5 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors:

$$T\left(\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

66. (5 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. Prove that $T : S_1 \to S_2$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one and onto.

Essay Questions

67. (5 points) Define an Elementary Matrix. Display the fundamental matrix multiply equation which summarizes a sequence of swap, combo, multiply operations, transforming a matrix A into a matrix B.

68. (5 points) Let V be a vector space and S a subset of V. Define what it means for S to be a **subspace** of V. The definition is sometimes called the **Subspace Criterion**, a theorem with three requirements, with the conclusion that S is a subspace of V.

69. (5 points) The null space S of an $m \times n$ matrix M is a subspace of $\mathbb{R}^n$. This is called the *Kernel Theorem*, and it is proved from the **Subspace Criterion**. Both theorems conclude that some subset is a subspace, but they have different hypotheses. Distinguish the Kernel theorem from the Subspace Criterion, as viewed from hypotheses.

70. (5 points) Least squares can be used to find the best fit line for the points $(1, 2)$, $(2, 2)$, $(3, 0)$. Without finding the line equation, describe how to do it, in a few sentences.

71. (5 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

72. (5 points) Display the equation for the Singular Value Decomposition (SVD), then cite the conditions for each matrix. Finish with a written description of how to construct the matrices in the SVD.

73. (5 points) State the **Spectral Theorem** for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the **spectral decomposition**.