MATH 2270-2 Sample Exam 2 S2012, revised April 1

1. (10 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Find a basis of vectors for each of the four fundamental subspaces.

2. (10 points) Assume $V = \mathbf{span}(\vec{v}_1, \vec{v}_2)$ with $\vec{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Find the Gram-Schmidt orthonormal vectors $\vec{q}_1, \vec{q}_2$ whose span equals V.

3. (10 points) Let Q be an orthonormal matrix. The normal equations for the system $Q\vec{x} = \vec{b}$ finds the least squares solution $\vec{v} = QQ^T\vec{b}$. The equations imply that $P = QQ^T$ projects $\vec{b}$ onto the span of the columns of Q. For the subspace $V = \mathbf{span}(\vec{v}_1, \vec{v}_2)$ in the previous problem, find matrix P. This matrix projects $\mathbb{R}^4$ onto V, while $I - P$ projects $\mathbb{R}^4$ onto $V^\perp$.

4. (10 points) Find the least squares best fit line $y = v_1 x + v_2$ for the points $(0, 1)$, $(2, 3)$, $(4, 4)$.

5. (5 points) Find the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

6. (10 points) Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$. Find all eigenpairs of A.

7. (10 points) Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$. Find all eigenpairs.

8. (15 points) Find an equation for the plane in $\mathbb{R}^3$ that contains the three points $(1,0,0)$, $(1,1,1)$, $(1,2,0)$.

9. (10 points) Suppose an $n \times n$ matrix A has all eigenvalues equal to 0. Show from the Cayley-Hamilton Theorem that A^n has all entries equal to 0.

10. (15 points) Prove the Cayley-Hamilton Theorem for 2×2 matrices with real eigenvalues. Write the characteristic equation as $\lambda^2 + c1\lambda = -c_2$, then substitute as in the Cayley-Hamilton theorem, arriving at the proposed equation $A^2 + c_1 A = -c_2 I$. Expand the left side:

$$A^2 + c_1 A = A(A + c_1 I) = A(A - (a+d)I) = -A\,\mathbf{adj}(A), \quad \mathbf{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Because $A\,\mathbf{adj}(A) = |A|I$ (the adjugate identity), then the right side of the preceding display simplifies to $-\det(A)I = -c_2 I$. This proves the Cayley-Hamilton theorem for 2×2 matrices: $A^2 + c_1 A = -c_2 I$.

11. (5 points) Suppose a 3×3 matrix A has eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

12. (10 points) Suppose a 3×3 matrix A has eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Find A.

13. (10 points) Assume A is 2×2 and Fourier's model holds:

$$A\left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 2c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find A.

14. (10 points) How many eigenpairs? (a) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

No new questions beyond this point.