MATH 2270-2 Final Exam Sample Problems Spring 2012

ANSWERS

Chapters 1 and 2

1. (5 points) Not on the final exam; from Cashen's course: Let A be a 2×2 matrix such that $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. Compute $A\begin{pmatrix}2\\2\end{pmatrix}$.

Answer:

$$A\left(\begin{array}{c}2\\2\end{array}\right) = 2\left(\begin{array}{c}1\\0\end{array}\right) = \left(\begin{array}{c}2\\0\end{array}\right)$$

2. (5 points) State (1) the definition of norm, (2) the Cauchy-Schwartz inequality and (3) the triangle inequality, for vectors in $\mathcal{R}^n$.

Answer:

(1) Norm of $\vec{v}$ equals $\|\vec{v}\| = \sqrt{\vec{v}^T\vec{v}}$; (2) $|\vec{a}\cdot\vec{b}| \le \|\vec{a}\|\|\vec{b}\|$; (3) $\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$.

3. (5 points) Suppose $A = B(C+D)E$ and all the matrices are $n \times n$ invertible. Find an equation for C.

Answer:

$AE^{-1} = BC + BD$ implies $C = B^{-1}(AE^{-1} - BD)$.

4. (5 points) Find all solutions to the system of equations

$$\begin{aligned}2w + 3x + 4y + 5z &= 1\\ 4w + 3x + 8y + 5z &= 2\\ 6w + 3x + 8y + 5z &= 1\end{aligned}$$

Answer:

Infinite solution case: $w = -1/2, x = -(5/3)t_1, y = 1/2, z = t_1$.

5. (5 points) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Show the details of two different methods for finding A^{-1}.

Answer:

The two methods are (1) $A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$ and (2) For $C =< A|I >$, then $\mathbf{rref}(C) =< I|A^{-1} >$.

6. (5 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

Let E_1 be the result of combo(1,2,-1/2) on I, and E_2 the result of combo(2,3,-2/3) on I. Then $E_2 E_1 A = U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$. Let $L = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$.

7. (5 points) Let Q be a 2×2 matrix with $QQ^T = I$. Prove that Q has columns of unit length and its two columns are orthogonal.

Answer:

First, $AB = I$ with both A, B square implies $BA = I$. So $Q^T Q = I$. Then $Q =< \vec{q}_1 | \vec{q}_2 >$ implies $Q^T Q = \begin{pmatrix} \vec{q}_1 \cdot \vec{q}_1 & \vec{q}_1 \cdot \vec{q}_2 \\ \vec{q}_2 \cdot \vec{q}_1 & \vec{q}_2 \cdot \vec{q}_2 \end{pmatrix}$. Relation $Q^T Q = I$ then implies orthogonality of the columns and that the columns have length one.

8. (5 points) True or False? If the 3×3 matrices A and B are triangular, then AB is triangular.

Answer:

False. Consider the decomposition $A = LU$ in a problem above.

9. (5 points) True or False? If a 3×3 matrix A has an inverse, then for all vectors $\vec{b}$ the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.

Answer:

True, $\vec{x} = A^{-1}\vec{b}$.

10. (5 points) Determine which values of k correspond to a **unique solution** for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer:

There is a unique solution for $\det(A) \neq 0$, which implies $k \neq 2$ and $k \neq 3$. Alternative solution: Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & 3-k & k-1 \end{pmatrix}$. Then (1) Unique solution for three lead variables, equivalent to the determinant nonzero for the frame above, or $(k-2)(3-k) \neq 0$; (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 2$.

Chapters 3, 4

11. (5 points) Explain how the **span theorem** applies to show that the set S of all linear combinations of the functions $\cosh x, \sinh x$ is a subspace of the vector space V of all continuous functions on $-\infty < x < \infty$.

Answer:

The span theorem says $\textbf{span}(\vec{v}_1, \vec{v}_2)$ is a subspace of V, for any two vectors in V. Choose the two vectors to be $\cosh x, \sinh x$.

12. (5 points) Write a proof that the subset S of all solutions $\vec{x}$ in $\mathcal{R}^n$ to a homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathcal{R}^n$. This is called the **kernel theorem**.

Answer:

(1) Zero is in S because $A\vec{0} = \vec{0}$; (2) If $A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{0}$, then $\vec{v} = \vec{v}_1 + \vec{v}_2$ satisfies $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$. So $\vec{v}$ is in S; (3) Let $\vec{v}_1$ be in S, that is, $A\vec{v}_1 = \vec{0}$. Let c be a constant. Define $\vec{v} = c\vec{v}_1$. Then $A\vec{v} = A(c\vec{v}_1) = cA\vec{v}_1 = (c)\vec{0} = \vec{0}$. Then $\vec{v}$ is in S. This completes the proof.

13. (5 points) Using the subspace criterion, write two hypotheses that imply that a set

S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Answer:

(1) If the zero vector is not in S, then S is not a subspace. (2) If two vectors in S fail to have their sum in S, then S is not a subspace. (3) If a vector is in S but its negative is not, then S is not a subspace.

14. (5 points) Report which columns of A are pivot columns: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer:

Zero cannot be a pivot column (no leading one in **rref**(A)). The other two columns are not constant multiples of one another, therefore they are independent and will become pivot columns in **rref**(A). Then: pivot columns =2,3.

15. (5 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

The augmented matrix has reduced row echelon form (last frame) equal to the matrix $\begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$. Then $x_1 = t_1, x_2 = 1, x_3 = 1$ is the general solution in scalar form. The partial derivative on t_1 gives the homogeneous solution basis vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $\vec{x}_h = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Set $t_1 = 0$ in the scalar solution to find a particular solution $\vec{x}_p = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

16. (5 points) Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

It is the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

17. (5 points) A 10×13 matrix A is given and the homogeneous system $A\vec{x} = \vec{0}$ is transformed to reduced row echelon form. There are 7 lead variables. How many free variables?

Answer:

Because $\vec{x}$ has 13 variables, then the rank plus the nullity is 13. There are 6 free variables.

18. (5 points) The rank of a 10×13 matrix A is 7. Find the nullity of A.

Answer:

There are 13 variables. The rank plus the nullity is 13. The nullity is 6.

19. (5 points) Determine independence or dependence for the list of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Answer:

Possible tests are the rank test, determinant test, pivot theorem. Let A denote the augmented matrix of the three column vectors. The determinant is 32, nonzero, so the vectors are independent. The pivot theorem also applies. The $\mathbf{rref}(A)$ is the identity matrix, so all columns are pivot columns, hence the three columns are independent. The rank test applies because the rank is 3, equal to the number of columns, hence independence.

20. (5 points) Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

Answer:

The first and fourth apply to the given functions, while the others apply only to fixed vectors.

21. (5 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer:

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{x} = \vec{0}$. Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.

Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, S is the nullspace of some matrix B, hence a subspace of $\mathcal{R}^3$. This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation $B\vec{x} = \vec{0}$.

22. (5 points) The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Answer:

Find $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 1 and 4.

23. (5 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Find the Gram-Schmidt orthonormal basis of S.

Answer:

Let $\vec{y}_1 = \vec{v}_1$ and $\vec{u}_1 = \frac{1}{\|\vec{y}_1\|}\vec{y}_1$. Then $\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Let $\vec{y}_2 = \vec{v}_2$ minus the shadow projection of $\vec{v}_2$ onto the span of $\vec{v}_1$. Then

$$\vec{y}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

Finally, $\vec{u}_2 = \frac{1}{\|\vec{y}_2\|}\vec{y}_2$. We report the Gram-Schmidt basis:

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

24. (5 points) Let A be an $m \times n$ matrix with independent columns. Prove that $A^T A$ is invertible.

Answer:

The matrix $B = A^T A$ has dimension $n \times n$. We prove that the nullspace of $B = A^T A$ is the zero vector.
Let $\vec{x}$ belong to $\mathcal{R}^n$. Assume $B\vec{x} = \vec{0}$, then multiply this equation by $\vec{x}^T$ to obtain $\vec{x}^T A^T A\vec{x} = \vec{x}^T\vec{0} = 0$. Therefore, $\|A\vec{x}\|^2 = 0$, or $A\vec{x} = \vec{0}$. If A has independent columns, then the nullspace of A is the zero vector, so $\vec{x} = \vec{0}$. We have proved that the nullspace of $B = A^T A$ is the zero vector.
An $n \times n$ matrix B is invertible if and only if its nullspace is the zero vector. So $B = A^T A$ is invertible.

25. (5 points) Let A be an $m \times n$ matrix with $A^T A$ invertible. Prove that the columns of A are independent.

Answer:

The columns of A are independent if and only if the nullspace of A is the zero vector. If you don't know this result, then find it in Strang's book, or prove it yourself.
Assume $\vec{x}$ is in the nullspace of A, $A\vec{x} = \vec{0}$, then multiply by A^T to get $A^T A\vec{x} = \vec{0}$. Because $A^T A$ is invertible, then $\vec{x} = \vec{0}$, which proves the nullspace of A is the zero vector. We conclude that the columns of A are independent.

26. (5 points) Let A be an $m \times n$ matrix and $\vec{v}$ a vector orthogonal to the nullspace of A. Prove that $\vec{v}$ must be in the row space of A.

Answer:

The fundamental theorem of linear algebra is summarized by `rowspace` $\perp$ `nullspace`. This relation implies `nullspace` $\perp$ `rowspace`, because for subspaces S we have $(S^\perp)^\perp = S$. The conclusion follows.

Chapter 5

27. (5 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

Answer:

$x_2 = \Delta_2/\Delta$, $\Delta_2 = \det\begin{pmatrix} -2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 3 & -2 \end{pmatrix} = 36$, $\Delta = \det(A) = 4$, $x_2 = 9$.

28. (5 points) Assume $A^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the inverse of the transpose of A.

Answer:

Compute $(A^T)^{-1} = (A^{-1})^T = \left(\begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}\right)^T = \begin{pmatrix} 2 & 0 \\ -6 & 4 \end{pmatrix}$.

29. (5 points) This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$..

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $C^{-1}B^2$.

$$B = \begin{pmatrix} 1 & -1 & ? & ? \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 2 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Answer:

The determinant of $C^{-1}B^2$ is $|B|^2/|C|$. Then $CB == \mathbf{adj}(B)B = |B|I$ implies $|C||B| = \det(|B|I) = |B|^4$. Because $|C| = |B|^3$, then the answer is $1/|B|$. Return to $CB = |B|I$ and do one dot product to find the value $|B| = 8$. We report $\det(C^{-1}B^2) = 1/|B| = 1/8$.

30. (5 points) Display the entry in row 3, column 4 of the adjugate matrix [or adjoint matrix] of $A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

Answer:

The answer is the cofactor of A in row 4, column 3 = $(-1)^7$ times minor of A in 4,3 = -2.

Chapter 6

31. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:

The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues, taken in the same order.

32. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (-r)(3-r)(r-2)^2$. The eigenvalues are $0, 2, 2, 3$. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a 3×3 determinant, which can be expanded by the cofactor method along column 3.

33. (5 points) The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$.
To save time, **don't find the eigenvector for** $\lambda = 0$.

Answer:

Because $A - 2I = \begin{pmatrix} -2 & -12 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ has last frame $B = \begin{pmatrix} 1 & 0 & -15/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then there is only one eigenpair for $\lambda = 2$, with eigenvector $\vec{v} = \begin{pmatrix} 15 \\ -2 \\ 2 \end{pmatrix}$.

34. (5 points) Find the two eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

Answer:

$$\left(-1+2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(-1-2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$$

35. (5 points) Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Answer:

The first and the last, because the test $A\vec{x} = \lambda\vec{x}$ passes in both cases.

36. (5 points) Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

Answer:

Both B and C have the same eigenvalues, because $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$. Further, both B and C are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix $B = \mathbf{diag}(-1, 1, 5)$. In this case, $A = 2I + 3B = \mathbf{diag}(2, 2, 2) + \mathbf{diag}(-3, 3, 15) = \mathbf{diag}(-1, 5, 17)$ and the eigenvalues of A are $-1, 5, 17$.

37. (5 points) Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

Answer:

Because $AP = PD$, then $D = P^{-1}AP$ is the diagonal matrix of eigenvalues, taken in the order determined by the eigenpairs $(3, \vec{v}_2), (1, \vec{v}_3), (4, \vec{v}_1)$. Then $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

38. (5 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Answer:

Compute $\mathbf{rref}(A - 3I) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This has rank 2, nullity 1. There is just one eigenvector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. No Fourier's model, not diagonalizable.

39. (5 points) Assume A is a given 4×4 matrix with eigenvalues 0, 1, $3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

Answer:

Such a matrix is diagonalizable, because of four distinct eigenvalues. Then $4B - 3I$ has the same eigenvalues for all matrices B similar to A. In particular, $4A - 3I$ has the same eigenvalues as $4D-3I$ where D is the diagonal matrix with entries $0, 1, 3+2i, 3-2i$. Compute $4D - 3I = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9+8i & 0 \\ 0 & 0 & 0 & 9-8i \end{pmatrix}$. The answer is $0, 1, 9+8i, 9-8i$.

40. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (r^2 + 6)(3 - r)(r - 2)^2$. The eigenvalues are $2, 2, 3, \pm\sqrt{6}i$. Determinant expansion is by the cofactor method along column 5. This reduces it to a 4×4 determinant, which can be expanded as a product of two quadratics. In detail, we first get $|A - rI| = (3 - r)|B - rI|$, where $B = \begin{pmatrix} 0 & -2 & -5 & 0 \\ 3 & 0 & -12 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$. So we have one eigenvalue 3, and we find the eigenvalues of B. Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer for the eigenvalues of A is 3 and the eigenvalues of B_1 and B_3. We report $3, \pm\sqrt{6}i, 2, 2$. It is also possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

41. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

Answer:

(1) $P = \begin{pmatrix} 13 & i & -i \\ 6 & 2 & 2 \\ -41 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$. (2) $AP = PD$ implies $A = PDP^{-1}$.

42. (5 points) Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

Answer:

Because the answer is the same for all matrices similar to A (that is, all $B = PAP^{-1}$) then it suffices to answer the question for diagonal matrices. We know A is diagonalizable, because it has distinct eigenvalues. So we choose D equal to the diagonal matrix with entries $2, 3, 4$. Compute $\frac{1}{3}D - 2I = \begin{pmatrix} \frac{2}{3} - 2 & 0 & 0 \\ 0 & \frac{3}{3} - 2 & 0 \\ 0 & 0 & \frac{4}{3} - 2 \end{pmatrix}$. Then the eigenvalues are $-\frac{4}{3}, -1, -\frac{2}{3}$.

43. (5 points) Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

Answer:

The proof depends on the identity $A - rI = PBP^{-1} - rI = P(B - rI)P^{-1}$ and the determinant product theorem $|CD| = |C||D|$. We get $|A - rI| = |P||B - rI||P^{-1}| = |PP^{-1}||B - rI| = |B - rI|$. Then A and B have exactly the same characteristic equation, hence exactly the same eigenvalues.

44. (5 points) Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Answer:

The characteristic polynomial is $\det(B - rI) = (2 - r)(5 - r)(5 - r)(3 - r)$. The eigenvalues are $2, 3, 5, 5$.

It is possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.

Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer is that B has eigenvalues equal to the eigenvalues of B_1 and B_3. These are quickly found by Sarrus' Rule applied to the two 2×2 determinants $|B_1 - rI| = (2-r)(5-r)$ and $|B_3 - rI| = r^2 - 8r + 15 = (5-r)(3-r)$.

45. (5 points) There are real 2×2 matrices A such that $A^2 = -4I$, where I is the identity matrix. Give an example of one such matrix A and then verify that $A^2 + 4I = 0$.

Answer:

Choose any matrix whose characteristic equation is $\lambda^2 + 4 = 0$. Then $A^2 + 4I = 0$ by the Cayley-Hamilton theorem.

46. (5 points) Let $Q = < \vec{q}_1 | \vec{q}_2 >$ be orthogonal and $D = \mathbf{diag}(\lambda_1, \lambda_2)$ a diagonal matrix. Prove that the 2×2 matrix $A = QDQ^T$ satisfies $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T$.

Answer:

Let $B = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T$. We prove $A = B$. First observe that both A and B are symmetric. Because the columns of Q form a basis of $\mathcal{R}^2$, it suffices to prove that $\vec{x}^T A = \vec{x}^T B$ for $\vec{x}$ a column of A. For example, take $\vec{x} = \vec{q}_1$. Then $\vec{x}^T A = (A^T \vec{q}_1)^T = (A\vec{q}_1)^T = \lambda_1 \vec{q}_1^T$. Orthogonality of Q implies $\vec{x}^T B = (B\vec{q}_1)^T = (\lambda_1 \vec{q}_1 \vec{q}_1^T \vec{q}_1 + \lambda_2 \vec{q}_2 \vec{q}_2^T \vec{q}_1)^T = \lambda_1 (\vec{q}_1 \cdot 1)^T = \lambda_1 \vec{q}_1^T$. Repeat for subscript 2 to complete the proof.

47. (5 points) A matrix A is positive definite if and only if $\vec{x}^T A \vec{x} > 0$ for nonzero $\vec{x}$. Which of these matrices are positive definite?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix}$$

Answer:

Only the second matrix. A useful test is positive eigenvalues. Another is principal determinants all positive.

48. (5 points) Let A be a real symmetric 2×2 matrix. Prove that the eigenvalues of A are real numbers.

Answer:

Begin with $A\vec{x} = \lambda \vec{x}$. Take the conjugate of both sides to get a new equation. Because the conjugate of a real matrix is itself, then the new equation looks like $A\vec{y} = \bar{\lambda}\vec{y}$ where $\vec{y}$ is the conjugate of $\vec{x}$. Formally, replace i by $-i$ in the components of $\vec{x}$ to obtain $\vec{y}$. Symbol $\bar{\lambda}$ is the complex conjugate of λ. Transpose this new equation to get $\vec{y}^T A = \bar{\lambda}\vec{y}^T$, possible because $A = A^T$. Taking dot products two ways gives $\vec{y} \cdot A\vec{x} = \lambda\vec{y} \cdot \vec{x} = \bar{\lambda}\vec{y} \cdot \vec{x}$. Because $\vec{y} \cdot \vec{x} = \|\vec{x}\|^2 > 0$, then we can cancel to get $\lambda = \bar{\lambda}$, proving the eigenvalue λ is real.

49. (5 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$.

Answer:

Start with the equation $r^2 - 6r + 8 = 0$ having roots $r = 2, 4$. Compute the eigenpairs $(2, \vec{v}_1)$, $(4, \vec{v}_2)$ where $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The two vectors are orthogonal but not of unit length. Unitize them to get $\vec{u}_1 = \frac{1}{\sqrt{2}}\vec{v}_1$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\vec{v}_2$. Then $Q =< \vec{u}_1|\vec{u}_2 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \mathbf{diag}(2, 4)$.

50. (5 points) Show that if B is an invertible matrix and A is similar to B, with $A = PBP^{-1}$, then A is invertible.

Answer:

The determinant product theorem applies to obtain $|A| = |B| \neq 0$, hence A is invertible.

51. (5 points) Give an example of a Jordan form of size 10×10 with exactly three different eigenvalues in five Jordan blocks.

Answer:

Let N_k be the nilpotent matrix (means a power of the matrix is the zero matrix) formed from the zero matrix of dimension $k \times k$ by making the superdiagonal entries all ones. Then any Jordan block can be written as a diagonal matrix plus a nilpotent matrix N_k. Define $J_1 = \mathbf{diag}(1, 1) + N_2$, $J_2 = \mathbf{diag}(1, 1) + N_2$, $J_3 = \mathbf{diag}(2, 2, 2) + N_3$, $J_4 = \mathbf{diag}(3, 3, 3) + N_3$. Then define $J = \mathbf{diag}(J_1, J_2, J_3, J_4)$. The eigenvalues of the Jordan form J are $1, 2, 3$ and there are 4 Jordan blocks.

52. (5 points) Prove that a Jordan block of size 3×3 has exactly one eigenvector.

Answer:

Write the matrix $A = \begin{pmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{pmatrix}$ where c is the eigenvalue of A. This is the general form of a Jordan block of dimension 3. We analyze $(A - cI)\vec{x} = \vec{0}$ in order to find all eigenvectors of A. This system in scalar form is $x_2 = 0, x_3 = 0, 0 = 0$. There is one free variable, therefore exactly one eigenvector.

53. (5 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

Answer:

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right)^T$$

54. (5 points) Describe, using a figure or drawing, the locations in the matrices U, V of the singular value decomposition $A = U\Sigma V^T$ which are consumed by the four fundamental subspaces of A. These are the nullspaces of A, A^T, the row space of A and the column space of A.

Answer:

$A =< colspace(A)|nullspace(A^T) > \Sigma < rowspace(A)|nullspace(A) >^T$. The dimensions of the spaces left to right are r, $m - r$, r, $n - r$, where A is $m \times n$ and r is the rank of A.

Chapter 7

55. (5 points) Give examples for a vertical shear and a horizontal shear in the plane. Expected is a 2×2 matrix A which represents the linear transformation.

Answer:

$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is a horizontal shear, $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ is a vertical shear

56. (5 points) Give examples for clockwise and counterclockwise rotations in the plane. Expected is a 2×2 matrix A which represents the linear transformation.

Answer:

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for $\theta > 0$ rotates clockwise and for $\theta < 0$ rotates counter clockwise.

57. (5 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors: Given a basis

$$T\left(\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

Answer:

$A\begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 5 \\ 4 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ can be solved for matrix A. The answer is $A = \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ 1 & \frac{1}{2} & -1 \\ -1 & \frac{5}{2} & -3 \end{pmatrix}$.

58. (5 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. Prove that $T : S_1 \to S_2$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one and onto.

Answer:

Suppose $\vec{x}$ is in the rowspace. The fundamental theorem of linear algebra says $\vec{x}$ is perpendicular to the nullspace. of A. So, if $\vec{x}_1, \vec{x}_2$ are vectors in the rowspace of A and $A\vec{x}_1) = A(\vec{x}_2$ then $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$. This implies $\vec{x} = \vec{x}_1 - \vec{x}_2$ belongs to the nullspace of A. But $\vec{x}$ is a linear combination of vectors in S_1, so it is in S_1, which is perpendicular to the nullspace. The intersection of V and $V^\perp$ is the zero vector, so $\vec{x} = \vec{0}$, which says $\vec{x}_1 = \vec{x}_2$, proving T is one-to-one.

The proof for onto is done by solving the equation $A\vec{x} = \vec{y}$ where $\vec{y}$ is any vector in the column space of A. We have to find $\vec{x}$ in S_1 that solves the equation. Select any $\vec{z}$ such that $\vec{y} = A\vec{z}$. Because the rowspace is perpendicular to the nullspace, then there are unique vectors $\vec{x}, \vec{u}$ such that $\vec{z} = \vec{x} + \vec{u}$, and $\vec{u}$ is in the nullspace while $\vec{x}$ is in the rowspace. Then $\vec{y} = A\vec{z} = A\vec{x} + A\vec{u} = A\vec{x} + \vec{0} = A\vec{x}$. We have solved the equation for $\vec{x}$ in S_1. The proof is complete.

59. (5 points) Assume singular value decomposition $A = U\Sigma V^T$ given by

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right)^T.$$

Find a formula for the pseudo-inverse, but don't bother to multiply out matrices.

Answer:

The definition is $A^+ = V\Sigma^+U^T$. Identifying the matrices from the formula gives $A^+ = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right)\begin{pmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T$.

Essay Questions

60. (5 points) Define an Elementary Matrix. Display the fundamental matrix multiply equation which summarizes a sequence of swap, combo, multiply operations, transforming a matrix A into a matrix B.

Answer:

An elementary matrix is the matrix E resulting from one elementary row operation (swap, combination, multiply) performed on the identity matrix I. The fundamental equation looks like $E_k \cdots E_2 E_1 A = B$, but this is not the complete answer, because the elementary matrices have to be explained, relative to the elementary operations which transformed A into B.

61. (5 points) Let V be a vector space and S a subset of V. State the **Subspace Criterion**, a theorem with three requirements, and conclusion that S is a subspace of V.

Answer:

The theorem can be found in Strang, although the naming convention might not be the same. Located on page 122 is a definition, which is often called the subspace criterion: (1) Zero is in S; (2) Sums of vectors in S are in S; (3) Scalar multiples of vectors in S are in S.

62. (5 points) The null space S of an $m \times n$ matrix M is a subspace of $\mathbb{R}^n$. This is called the *Kernel Theorem*, and it is proved from the *Subspace Criterion*. Both theorems conclude that some subset is a subspace, but they have different hypotheses. Distinguish the Kernel theorem from the Subspace Criterion, as viewed from hypotheses.

Answer:

The distinction is that the kernel theorem applies only to fixed vectors, that is, the vector space $\mathcal{R}^n$, whereas the subspace criterion applies to any vector space.

63. (5 points) Least squares to find the best fit line for the points $(1, 2)$, $(2, 2)$, $(3, 0)$. Without finding the line equation, describe how to do it, in a few sentences.

Answer:

Find a matrix equation $A\vec{x} = \vec{b}$ using the line equation $y = v_1 x + v_2$ where $\vec{x} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then solve the normal equation $A^T A\vec{v} = A^T\vec{b}$. A full solution is expected, with a formula for A. But don't solve the normal equation.

64. (5 points) Display the equation for the pseudo inverse of A, then define and document each matrix in the product.

Answer:

See section 7.3. Yes, you are expected to know a formula for A^+.

65. (5 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

Answer:

Part 1. The dimensions are $n - r, r, rm - r$ for nullspace(A), colspace(A), rowspace(A), nullspace(A^T). Part 2. The orthogonality relation is rowspace $\perp$ nullspace, for both A and A^T. A full statement is expected, not the brief one given here.

66. (5 points) Display the equation for the Singular Value Decomposition (SVD), then cite the conditions for each matrix.

Answer:

Let r be the rank of an $m \times n$ matrix A. Then $A = U\Sigma V^T$, where $A\vec{v}_i = \sigma_i \vec{u}_i$, $U =< \vec{u}_1 | \cdots | \vec{u}_n >$, $V =< \vec{v}_1 | \cdots | \vec{v}_m >$ are orthogonal and $\Sigma = \textbf{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0 >$.