Math 310, Lesieutre
Exam 2 Practice
November 11, 2015

Name: ______________________________

This is a version of an old exam replacing some topics we didn't cover yet. Solutions will be posted over the weekend.

Please let me know if you spot any mistakes in my solutions.

1. *Consider the matrix* $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$.

   (a) *What are the eigenvalues of A?*

   We need to compute the characteristic polynomial.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix} = (3-\lambda)(-\lambda) - (1)(-2) \\ &= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2). \end{aligned}$$

   Hence the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

   (b) *What are the eigenvectors of A?*

   We have

$$A - 1 \cdot I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix},$$

   for which $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the nullspace.

   Similarly,

$$A - 2 \cdot I = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix},$$

   for which $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in the nullspace.

   (c) *Find a diagonalization of the matrix A.*

   To find this, we just stick the eigenvectors in as the columns of P.

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

   These satisfy $A = PDP^{-1}$.

   (d) *Compute the matrix* A^5.

   Well,

$$\begin{aligned} A^5 = PD^5P^{-1} &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix} \left(\frac{1}{-1} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 64 \\ 1 & 32 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 63 & -62 \\ 31 & -30 \end{bmatrix}. \end{aligned}$$

2. *The matrix B has reduced echelon form U, where*

$$B = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & -1 & -3 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) *Give a basis for* $\operatorname{Row} B$. *What is the dimension?*

That's easy: just take the nonzero rows of U.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

There are three of them, so the dimension is 3.

(b) *Give a basis for* $\operatorname{Col} B$. *What is the dimension?*

We see that the pivots of U are in columns 1, 2, and 4. So a basis for the column space is given by colmuns 1, 2, and 4 of the matrix B:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

There are three, so the dimension is 3 again. This is no coincidence, of course: the dimension of the row space is always equal to the dimension of the column space.

(c) *Give a basis for* $\operatorname{Nul} B$. *What is the dimension?*

There are two free variables when solving $B\mathbf{x} = \mathbf{0}$, x_3 and x_5. Call those parameters s and t. The general solution is

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} = \begin{bmatrix} -s+t \\ -s-t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

A basis for the nullspace is given by the two vectors

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

It is two dimensional.

(d) *For what values of a and b does the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & a & 2 & 2 \\ 0 & 0 & 0 & b & 2 \end{bmatrix}$$

have rank 2?

Hmm, let's think about this. If a is not zero, then there are three pivots, no matter what the value of b is (the third pivot column is column 4 is b is 0, and column 5 otherwise). If a is zero, there's some hope. If $a = 0$ and $b = 2$, then the third row is equal to the second, and so the row space is only two dimensional, and the rank is 2. If $a = 0$ and b is anything other than 2, then the rows are linearly independent. So the only way to get the rank to be 2 is of $a = 0$ and $b = 2$.

3. *Let* $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *be the standard basis for* $\mathbb{R}^2$.

(a) *Show that* $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ *is another basis for* $\mathbb{R}^2$.

This is two vectors in $\mathbb{R}^2$, so to show it's a basis, it's enough to show that they're linearly independent. Does $A\mathbf{x} = \mathbf{0}$ have any solutions? Well, let's see.

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -1/2 & 0 \end{array}\right].$$

This is echelon form and there are two pivots, so no free variables. That means it's a basis.

(b) *Write down the change of basis matrix* $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}$ *from* $\mathcal{B}$ *to the standard basis* $\mathcal{E}$.

We have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{E}}^{-1}\mathcal{P}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}.$$

(c) *Find the $\mathcal{B}$-coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of the vector* $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We have

$$\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1} = \frac{1}{(2)(-1)-(1)(-1)} \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}.$$

Then

$$[\mathbf{v}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$$

4. *Consider the matrix*

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) *What is the dimension of the eigenspace corresponding to the eigenvalue* 1*? (You do not need to compute a basis.)*

This eigenspace with eigenvalue 1 is the nullspace of $C - 1I$, and that we know how to find.

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

This is already echelon form, and there is only one free variable x_1. So the dimension of the eigenspace is 1.

(b) *What is the dimension of the eigenspace corresponding to the eigenvalue* 2*? (You do not need to compute a basis.)*

This time we want the nullspace of $C - 2I$.

$$\left[\begin{array}{ccc|c} -1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This is echelon form. There's a single free variable x_2. So there's a 1-dimensional nullspace of $C - 2I$, and the eigenspace is 1-dimensional. ‘

(c) *Explain why the matrix C is not diagonalizable.*

We had the eigenvalue 2 show up twice, but there's only one independent eigenvector for it. That's not enough to form the matrix P in diagonalization; we would need to have two eigenvectors with $\lambda = 2$.

5. (a) *Find an LU decomposition of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 3 \\ 2 & 2 & 3 & 4 \end{bmatrix}.$$

We have to row reduce until we get to echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 3 \\ 2 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

That's echelon form! So $U = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. The row operations we did were:

- Subtract 3 row 1 from row 2.
- Subtract 2 row 1 from row 3.
- Subtract -2 row 2 from row 3.

So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) *Use your LU decomposition to find a solution to* $A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

We want $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 4 \\ 2 & -2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & -2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

So

$$\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Now we want $U\mathbf{x} = \mathbf{y}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

There are multiple solutions (as there must be, since A is 3×4). We're only asked to find *a* solution, so we can just plug in 0 for the free variable x_4. This gives

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

6. *Consider the matrix*

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

(a) *Compute the determinant of* A *using row reduction.*

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$

We've reached echelon form. The determinant is the product of the pivots, which is -9.

(b) *Compute the determinant of* A *using cofactor expansion.*

This is going to be easiest using expansion along the bottom row.

$$\det A = -3 \det \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = -3(3) = -9.$$

(c) *Use Cramer's rule to solve*

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for x_3. (You do not need to solve for x_1 and x_2.)

We have

$$x_3 = \frac{\det A_3(\mathbf{b})}{\det A} = \frac{\det \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix}}{\det A} = \frac{1}{-9} \det \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \frac{1}{-9}(-6) = \frac{2}{3}$$