Theory of Equations

Main Theorems and Methods

The topics apply to root-finding and factorization of any polynomial.

Example. Solve $x^3 + 3x^2 + 2x = 0$ for $x = 0, -1, -2$ and then factor the polynomial into linear factors x, $(x + 1)$, $(x + 2)$.

Quadratic $ax^2 + bx + c = 0$, http://en.wikipedia.org/wiki/Quadratic_equation

Long Division Algorithm, http://en.wikipedia.org/wiki/Polynomial_long_division

Descartes' Rule of Signs, http://en.wikipedia.org/wiki/Descartes'_rule_of_signs

Factor Theorem and Root Theorem, http://en.wikipedia.org/wiki/Factor_theorem

Sum and Product of the Roots, http://en.wikipedia.org/wiki/Vieta's_formulas

Rational Root Theorem, http://en.wikipedia.org/wiki/Rational_root_theorem

Quadratic Equations

The second order polynomial equation

$$ax^2 + bx + c = 0$$

can be solved by the following methods, illustrated in detail in the **WikiHow** link http://www.wikihow.com/Factor-Second-Degree-Polynomials-(Quadratic-Equations)

Inverse FOIL Method, e.g., $x^2 + 3x + 2 = (x + 2)(x + 1)$ based on guessing linear factors $(x + 2), (x + 1)$, then test with FOIL expansion.

Difference of Squares, using identity $A^2 - B^2 = (A - B)(A + B)$.

Complete the Square, using identity $u^2 + Bu = \left(u + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$.

Quadratic Formula, $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

Cubic Equations

Cardano provided formulas in 1545 to solve any cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

The formulas can be found at the link https://proofwiki.org/wiki/Cardano's_Formula, although computer algebra systems provide an easier interface.

A number of simple higher order equations can be solved by using the theory of equations. An illustration:

$$x^3 - 3x - 2 = 0.$$

First, rational roots are tried from the list $\pm 1, \pm 2$ predicted by the **Rational Root Theorem**. The first root found is $x = -1$. The **Factor Theorem** implies $x - (-1)$ is a factor. Then the **Division Algorithm** applies: $\dfrac{x^3 - 3x - 2}{x + 1} = x^2 - x - 2$. This quadratic is factored via **Inverse FOIL**: $(x + 1)(x - 2)$. Then the final factorization is

$$x^3 - 3x - 2 = (x + 1)(x + 1)(x - 2).$$

The roots are $-1, -1, 2$.