Linear Transformation

A linear transformation is a function $\boldsymbol{T}$ defined on a vector space $\boldsymbol{V}$ with range in a vector space $\boldsymbol{W}$ satisfying the rules
(a) $\boldsymbol{T}(\mathrm{v}_1 + \mathrm{v}_2) = \boldsymbol{T}(\mathrm{v}_1) + \boldsymbol{T}(\mathrm{v}_2)$
(b) $\boldsymbol{T}(k\mathrm{v}_1) = k\boldsymbol{T}(\mathrm{v}_1)$.

Theorem 1 (Matrix of T)
Assume $\boldsymbol{V} = \boldsymbol{R}^n$ and $\boldsymbol{W} = \boldsymbol{R}^m$. Then $\boldsymbol{T}$ is represented as a matrix multiply

$$\boldsymbol{T}(\mathrm{x}) = \boldsymbol{A}\mathrm{x}$$

where $\boldsymbol{A}$ is the $n \times m$ matrix whose columns are given in terms of the identity matrix $\boldsymbol{I}$ and function $\boldsymbol{T}$ by the formula

$$\mathrm{col}(\boldsymbol{A}, j) = \boldsymbol{T}(\mathrm{col}(\boldsymbol{I}, j)), \quad j = 1, \ldots, n.$$

Definition: A basis of a vector space $\boldsymbol{V}$ is a set of vectors $\mathrm{v}_1, \ldots, \mathrm{v}_n$ such that every vector v in $\boldsymbol{V}$ can be uniquely written as a linear combination of $\mathrm{v}_1, \ldots, \mathrm{v}_n$. Briefly, the vectors *span* V and are *independent*.

Theorem 2 (Representation of T)
Every basis $\{\mathrm{v}_1, \ldots, \mathrm{v}_n\}$ of $\boldsymbol{V}$ gives a relation

$$\boldsymbol{T}\left(\sum_{j=1}^{n} c_j \mathrm{v}_j\right) = \sum_{j=1}^{n} c_j \mathrm{w}_j, \quad \text{where} \quad \mathrm{w}_j = \boldsymbol{T}(\mathrm{v}_j).$$