Math 2280 Extra Credit Problems
Chapter 9
S2015

Submitted work. Please submit one stapled package per chapter. Kindly label problems [Extra Credit]. Label each problem with its corresponding problem number, e.g., Xc1.2-4. Please attach this printed sheet to simplify your work.

Chapter 9: 9.1, 9.2, 9.3 – Periodic Functions and Fourier Series

Problem Xc9.0-1. (Trigonometric Identities and Integrals)
(a) Use the trigonometric identity \( \cos(a + b) = \cos a \cos b - \sin a \sin b \) to derive the trigonometric identity
\[
\cos(mx) \cos(nx) = \frac{1}{2} (\cos((m + n)x) + \cos((m - n)x)).
\]
(b) Show the details for integrating \( \cos(mx) \cos(nx) \) for nonnegative integers \( m \neq n \) over \(-\pi \leq x \leq \pi\).
(c) Derive the trigonometric identity \( \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \) from the trigonometric identities \( \cos(a + b) = \cos a \cos b - \sin a \sin b \) and \( \cos^2 \theta + \sin^2 \theta = 1 \).
(d) Integrate \( \cos^2 nx \) for integers \( n = 0, 1, 2, 3, \ldots \) over \(-\pi \leq x \leq \pi\). Explain geometrically why there are two different answers.

Problem Xc9.0-2. (Orthogonality)
Two vectors \( \vec{A}, \vec{B} \) are said to be orthogonal if their dot product is zero. For vectors of dimension \( n \), this means \( a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = 0 \).
(a) The equation \( \int_0^1 f(x)g(x)dx = 0 \) can be viewed as the Riemann sum is approximately zero. Argue that this means \( \vec{A} \cdot \vec{B} = 0 \) to so many decimal places, where \( \vec{A} \) and \( \vec{B} \) are vector samples of \( f \) and \( g \) represented in the Riemann sum \( h \sum_{j=1}^{n} f(jh)g(jh), \ h = \frac{1}{n} \).
(b) Prove the orthogonality relation below from the standard one for the trigonometric system on \(-\pi \leq x \leq \pi\), by a change of variables. The method leads to six orthogonality relations for the trigonometric system \( \{\cos(m\pi x/T)\}_{m=0}^\infty, \{\sin(n\pi x/T)\}_{n=1}^\infty \) on \(-T \leq u \leq T\).
\[
\int_{-T}^{T} \cos(m\pi u/T) \cos(n\pi u/T) du = \begin{cases} 0 & m \neq n, \\ T & m = n. \end{cases}
\]

Problem Xc9.1-1. (Periodic Functions)
(a) Find the period, amplitude and frequency of \( \sin^2(4x) \).
(b) Let \( f \) be periodic of period 1 and on \( 0 \leq x \leq 1 \) \( f(x) = f_0(x) \), where \( f_0(x) = 1 \) on \( 0 \leq x < 1/2, f_0(x) = 0 \) on \( 1/2 \leq x < 1, f_0(1) = 0 \). Graph \( f \) on \(-2 \leq x \leq 3\).

Problem Xc9.1-8. (Sums of Periodic Functions)
(a) Find the period of \( \cos x + \cos 3x \).
(b) Find the period of \( e^{2\cos 2x} \).
Problem Xc9.1-8. (Periodic Functions)
Explain why \( \cos x + \cos 3\pi x \) is not periodic.

Problem Notes. One explanation uses independence of functions. Another explanation analyzes the number of solutions of the equation \( \cos x + \cos 2\pi x = 2 \). Expected in this case is a graphic and a mathematical argument [use \(-2 \leq f(x) \leq 2\)]

Problem Xc9.1-18. (Change of Variables)
(a) Prove that \( f(x) \) continuous and \( T \)-periodic implies \( \int_0^T f(x)dx = \int_{nT}^{nT+T} f(u)du \).

(b) Assume \( f(x) \) is 2\( \pi \)-periodic and continuous. Prove that \( F(x) = \int_0^x f(u)du \) is 2\( \pi \)-periodic if and only if \( \int_0^{2\pi} f(x)dx = 0 \).

Problem Xc9.1-20. (Floor Function)
The greatest integer function or staircase function is represented using a library function \( \text{floor}(x) \), available in most computer mathematical workbenches, including MATLAB and Maple. Don’t confuse \( \text{floor} \) with \( \text{trunc} \) – they are different functions!

(a) Plot \( \text{floor}(x) \) and \( x - \text{floor}(x) \) in MATLAB or Maple on \(-3 \leq x \leq 5\). Programs Excel and OpenOffice can also be used to make the plot. The \( \text{floor} \) function in Excel is \( x \rightarrow \text{FLOOR}(x,1) \).

(b) Argue from the graphic that \( x - \text{floor}(x) \) is periodic of period 1.

(c) Define \( g(x,T) = x - T \text{ floor}((x + T/2)/T) \). Show mathematically that \( g(x) = x \) on \( |x| < T/2 \). That the triangular wave \( g \) is \( T \)-periodic can be seen from its graphic.

(d) Plot the 2-periodic triangular wave defined by \( f(x) = |x - 2 \text{floor}((x + 1)/2)| \).

Problem Notes: The function \( f \) in (d) equals \( |g(x,T)| \), where \( T = 2 \) and \( g \) is defined in (c) above. The function \( a|g(x,T)| \) is called a triangular wave of height \( a \) and period \( T \). It is the composition of \( u \rightarrow a|u| \) and \( x \rightarrow g(x,T) \).

Problem Xc9.2-5. (Fourier Series Partial Sum Plots)
(a) Plot on \(-2\pi \leq x \leq 3\pi \) the partial sums \( s_2(x), s_6(x), s_{12}(x), s_{20}(x) \) of the Fourier series for the sawtooth wave \( f \) constructed from \( f_1(x) = |x| \) on \( |x| \leq \pi \):

\[
s_N(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{N} \frac{1}{(2m + 1)^2} \cos(2mx + x).
\]

The four graphics show the convergence of the partial sums to the limiting Fourier series. This is an example of a filmstrip of 4 graphics.

(b) Explain what happens in the graphic of (a) at points of discontinuity of \( f' \).

Problem Xc9.2-5a. (Fourier Series Partial Sum Plots)
(a) Plot on \(-2\pi \leq x \leq 3\pi \) the partial sums \( s_2(x), s_6(x), s_{12}(x), s_{20}(x) \) of the Fourier series for the sawtooth wave \( f \) constructed from \( f_1(x) = (\pi - x)/2 \) on \( 0 < x \leq 2\pi \):

\[
s_N(x) = \sum_{n=1}^{N} \frac{1}{n} \sin(nx).
\]

The four snapshots show the convergence of the partial sums to the limiting Fourier series.

(b) Explain what happens in the graphic of (a) at points of discontinuity of \( f \).

(c) Illustrate Gibb’s phenomenon. In particular, graph \(|f(x) - s_N(x)|\) on \(-2\pi \leq x \leq 3\pi \), for \( N = 6, 12, 20 \). Then estimate the jump at points of discontinuity of \( f' \).

Answer: About 1.25.

Problem Xc9.2-8. (Fourier Series Computation and Graphics)
Let \( f \) be the 2\( \pi \)-periodic extension of the rectified cosine wave \( f_0(x) = |\cos x| \) on \( |x| \leq \pi \).

(a) Draw a graphic of \( f(x) \) on \(-4\pi \leq x \leq 5\pi \).

(b) Show the derivation details for the Fourier series \( \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2 - 1} \cos 2mx \).

(c) Plot the Fourier series on \(-2\pi \leq x \leq 2\pi \). Explain why it differs from the plot of \( f(x) \) on the same interval.
Problem Xc9.2-15. (Fourier Series Computation)
Show the derivation details for the Fourier coefficients of \( f(x) \) constructed from \( f_2(x) = e^{-|x|} \) on \( |x| \leq \pi \). The Fourier series is
\[
\frac{e^{\pi} - 1}{\pi e^{\pi}} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{e^{\pi} + (-1)^{m+1}}{m^2 + 1} \cos mx.
\]

Problem Xc9.0-3. (Even and Odd Functions)
(a) Define even function and odd function. Such functions don’t have to be continuous, but they must be defined for all \( x \).
(b) Show the mathematical details in the derivation of the result \((\text{Even})(\text{Odd})=\text{Odd}\).
(c) Prove by a \( u \)-substitution that \( \int_{-p}^{p} f(x)dx = 2 \int_{0}^{p} f(x)dx \) for an even continuous function \( f \) and \( \int_{-p}^{p} g(x)dx = 0 \) for an odd continuous function \( g(x) \).

Problem Xc9.3-7. (Fourier Series Arbitrary Period)
(a) Define \( f \) to be the periodic extension of period 4 of the base function \( f_0(x) = 1 - x \) on \( 0 \leq x \leq 2 \), \( f_0(x) = -1 - x \) on \( -2 \leq x \leq 0 \). Plot \( f(x) \) on \( -8 \leq x \leq 6 \).
(b) Show the derivation details for the Fourier series of \( f(x) \):
\[
\frac{4}{\pi} \sum_{n=1}^{\infty} \sin(n\pi x) \frac{1}{2n}.
\]

Problem Xc9.3-32. (Failure of Term-by-Term Differentiation)
Show that the Fourier series \( \sum_{n=1}^{\infty} \frac{\sin nx}{n} \) of the sawtooth wave \( f \) cannot be differentiated term-by-term to obtain the Fourier series of \( f' \).

Problem Xc9.3-34. (Term-by-Term Integration)
Integrate the Fourier series of the triangular wave \( f \) constructed from \( f_0(x) = x \) on \( |x| \leq 1 \), in order to find the Fourier series of the parabolic wave \( g \) constructed from \( g_0(x) = x^2 \) on \( |x| \leq 1 \).

Chapter 9: 9.3, 9.4 – Fourier Series Methods

Problem Xc9.0-4. (Periodic Extensions)
Lemma 1. The function \( tw(x) = x - \text{floor}(x + 1/2) \) is a triangular wave of period 1 with shape \( x \) on \( |x| < 1/2 \).

Lemma 2. Given \( f_0(x) \) defined on \( |x| \leq T/2 \), then \( f(x) = f_0(Ttw(x/T)) \) is the \( T \)-periodic extension of \( f_0(x) \) from \( |x| \leq T/2 \) to \( -\infty < x < \infty \).

Assume Lemmas 1 and 2 for this problem.
(a) Plot \( f_1(x) = 3tw(x/3) \) on \( -6 \leq x \leq 6 \). Document its period on the graphic.
(b) Define \( f_2(x) = |\cos(0.5\pi tw(2x/\pi))| \). Make a plot on \( -2\pi \leq x \leq 3\pi \). Document its period on the graphic.

Problem Xc9.0-5. (Even and Odd Periodic Extensions)
Definition. Define \( \text{signum}(x) = \begin{cases} \frac{x}{|x|} & x \neq 0, \\ 0 & x = 0. \end{cases} \)
There is no agreement in literature how to define \( \text{signum}(0) \). Here, \( \text{signum}(x) \) takes on only the values 1, -1 and 0.
(a) Let \( p = 2 \) and define \( g_1(x) = x^2 \) on \( 0 \leq x \leq p \). Let \( g_2(x) = \text{signum}(x)g_1(|x|) \) be the odd extension of \( g_1 \) to \( |x| \leq p \). Let \( T = 2p \). Define \( f_3(x) = g_2(Ttw(x/T)) \) to be the odd extension of \( g_2(x) \) from \( |x| \leq p \) to \( -\infty < x < \infty \). Plot \( f_3 \) on \( |x| \leq 5 \). This sequence of formulas works in general, for any \( p \) and any \( g_1 \) (no justification requested).
(b) Let \( p = 2 \) and define \( h_1(x) = x^2 \) on \( 0 \leq x \leq p \). Let \( h_2(x) = h_1(|x|) \) be the even extension of \( h_1 \) to \( |x| \leq p \). Let \( T = 2p \). Define \( h_3(x) = h_2(T \text{tw}(x/T)) \) to be the even extension of \( h_2 \) from \( |x| \leq p \) to \( -\infty < x < \infty \). Plot \( h_3 \) on \( |x| \leq 5 \). This sequence of formulas works in general, for any \( p \) and any \( h_1 \) (no justification requested).

**Problem Xc9.0-6. (Dirichlet Kernel Identity)**

Establish by trigonometric identity methods the formula \([\text{the right side is called Dirichlet's Kernel}]\)

\[
\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin \left( nx + \frac{x}{2} \right)}{2 \sin \left( \frac{x}{2} \right)}.
\]

**Hint:** Cross multiply by \( 2 \sin(x/2) \). Expand terms using a trigonometric identity, which produces a telescoping sum.

**Problem Xc2.4-7. (Half-Range Expansions)**

(a) Find a simple algebraic formula for the even \( \pi \)-periodic extension of \( f_0(x) = \cos x \) on \( 0 \leq x \leq \pi/2 \).

(b) Find the Fourier coefficients for the half-range expansion of \( f_0(x) = \cos x \) on \( 0 \leq x \leq \pi/2 \).

**Problem Xc9.4-15. (Half-Range Sine Expansion)**

Find the Fourier coefficients for the half-range sine series expansion of \( e^x \) on \( 0 \leq x \leq 1 \).

**Problem Xc9.4-6. (Complex Fourier Series)**

Find the complex form of the Fourier series for \( \sin 3x \) without evaluating any trigonometric integrals.

**Hint:** Use \( \sin u = \frac{1}{2i} (e^{iu} - e^{-iu}) \).

**Problem Xc9.4-11. (Series Identities)**

Let \( x = 0 \) in the complex Fourier series expansion of \( e^x \) in order to prove the formula

\[
\frac{2\pi}{e^\pi - e^{-\pi}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + 1}.
\]

**Chapter 9: 9.5 – One Dimensional Heat Equation**

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**Problem Xc9.6-13. (Nonhomogeneous Heat Equation)**

Consider the one-dimensional heat conduction problem

\[
\begin{align*}
  u_t &= u_{xx}, \quad 0 \leq x \leq \pi, \quad t > 0, \\
  u(0, t) &= 100, \\
  u(\pi, t) &= 50, \\
  u(x, 0) &= f(x).
\end{align*}
\]

Assume \( f(x) = 33x \) on \( 0 < x \leq \pi/2 \), \( f(x) = 33\pi - 33x \) on \( \pi/2 < x < \pi \). Find a solution formula for the temperature \( u(x, t) \).

**Problem Xc9.6-3. (Heat Conduction in an Insulated Bar)**

Consider the one-dimensional heat conduction problem

\[
\begin{align*}
  u_t &= u_{xx}, \quad 0 \leq x \leq 1, \quad t > 0, \\
  u_x (0, t) &= 0, \\
  u_x (1, t) &= 0, \\
  u(x, 0) &= \cos \pi x
\end{align*}
\]

Find a solution formula for the temperature \( u(x, t) \) at location \( x \) along the bar at time \( t \). Hint: Don’t integrate!

**Remark.** Asmar’s matching problem 3.6-3 has a piecewise example, using \( u(x, 0) = f(x) \). See the maple advice for problem 9.5-13, to handle that case.
Chapter 9: 9.6 – One Dimensional Wave Equation

Problem Xc9.6-1. (Wave Equation)
Derive the equation \( u_{tt} = 10u_{xx} \) for the vibrations of a stretched homogeneous string with linear density \( \rho = 0.001 \text{ kg/m} \) and tension \( \tau = 100 \text{ N} \), with no forces other than the tension. State all assumptions used to obtain the model. Make the presentation brief, by referencing a textbook for derivation details and results.

Problem Xc9.6-9a. (Separation of Variables)
Solve \( u_{tt} = u_{xx}, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = \sin \pi x, \quad t \geq 0, \quad 0 \leq x \leq 1 \). The model is for a guitar string of unit length.

Problem Xc9.6-9b. (Filmstrip Plots)
Plot partial sums of the answer to the previous problem,
\[
\begin{align*}
&u(x,t) = 1/\pi \sin(\pi x) \sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^4(2m+1)^3} \sin(2m\pi x + \pi x) \cos(2m\pi t + \pi t),
&\text{at } t = 0, 1, 2, 3.
\end{align*}
\]
Choose the number of series terms for the four graphics by making the first graphic match \( x(1-x) \) on \( 0 \leq x \leq 1 \). This filmstrip has 4 frames, each frame corresponding to a time \( t \). A frame has graph window \( 0 \leq x \leq 1, \quad a \leq u \leq b \) (you must choose \( a, b \)).

Problem Xc9.6-9c. (Surface Plot)
Plot a specific partial sum of the answer
\[
\begin{align*}
&u(x,t) = 1/\pi \sin(\pi x) \sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^4(2m+1)^3} \sin(2m\pi x + \pi x) \cos(2m\pi t + \pi t)
&\text{on the domain } 0 \leq x \leq 1, \quad 0 \leq t \leq 4.
&\text{Use all features possible of the 3D graphics program in order to produce the best plot with fine accuracy, view and colors.}
\end{align*}
\]

Problem Xc9.6-13. (Damped Vibrations of a String)
Solve the problem
\[
\begin{align*}
&u_{tt}(x,t) + u_t(x,t) = u_{xx}(x,t), \\
&u(0,t) = 0, \\
&u(\pi,t) = 0, \\
&u(x,0) = \sin x, \\
&u_t(x,0) = 0.
\end{align*}
\]

Problem Xc9.6-1. (Vibrating Finite String)
Solve the wave equation \( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \) with boundary and initial conditions \( u(0,t) = u(L,t) = 0, \quad u(x,0) = \frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{2} \sin \frac{2\pi x}{L} + 2 \sin \frac{7\pi x}{L}, \quad u_t(x,0) = 0, \quad 0 < x < L, \quad t \geq 0 \). Use the series formula \( u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{nx}{L} \cos \frac{nc \pi t}{L} \).

References: Edwards-Penney section 9.6 (2280 textbook) and Asmar’s text, PDE and BVP, section 1.2.

Problem Xc9.6-2. (Loudness)
The fraction of the loudness associated with the fundamental tone (\( b_1 \)-term in the series) is the quotient
\[
F_1 = \frac{\left(\frac{n^2 b_1^2}{\pi^2}\right)}{\sum_{k=1}^{\infty} \frac{k^2 b_k^2}{\pi^2}}
\]
Find an approximation to the percentage 100\( F_1 \).

References: ProbXc9.6-1. The discussion of music in E&P includes a derivation of the formula for the percentage loudness 100\( F_n \).
Chapter 9: 9.6 – d’Alembert’s Method

Problem Xc9.6-15. (d’Alembert’s Solution)
Consider the problem

\[ u_{tt} = u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0, \]
\[ u(0,t) = 0, \]
\[ u(1,t) = 0, \]
\[ u(x,0) = f(x), \]
\[ u_t(x,0) = 0. \]

Assume \( f(x) = 4x \) on \( 0 \leq x \leq 0.25 \), \( f(x) = 2 - 4x \) on \( 0.25 < x \leq 0.5 \), \( f(x) = 0 \) on \( 0.5 < x \leq 1 \).

(a) Find a solution formula for \( u(x,t) \) using d’Alembert’s method.
(b) Plot a 3-frame filmstrip of the string shape at times \( t = 0, 0.25, 0.5 \).

# EXAMPLE. Let \( f(x)=4x \) on \([0,.25]\), \( f(x)=2-4x \) on \([.25,.5]\), \( f(x)=0 \) otherwise
# Asmar 3.4-15, D’Alembert’s solution of the wave equation, \( f=pulses, g=0 \)
pulse:=(x,a,b)->piecewise(x<a,0,x<b,1,0);
f:=x->4*x*pulse(x,0,1/4)+(2-4*x)*pulse(x,1/4,1/2);
#plot(f(x),x=0..1);
F:=x->piecewise(x<0,-f(-x),f(x)); # Odd extension of \( f(x) \)
plot(F(x),x=-1..1);
u:=(x,t)->(1/2)*(F(x+t)+F(x-t));
#plot(u(x,0.7),x=-2..2);
plots[animate]( plot, [u(x,t),x=-3..3], t=0..1.5, trace=0, frames=50 );

Problem Xc9.6-18. (Energy Conservation and d’Alembert’s Solution)
Define

\[ E(t) = \frac{1}{2} \int_0^L \left( u_t^2(x,t) + c^2 u_x^2(x,t) \right) \, dx. \]

Prove the energy conservation law, which says that the energy during free vibrations of a string is constant for all time.

Problem Notes. Show \( dE/dt = 0 \).

End of extra credit problems chapter 9.