Name ____________________________ Class Time _____

Math 2280 Extra Credit Problems
Chapter 9
S2015

Submitted work. Please submit one stapled package per chapter. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., Xc1.2-4. Please attach this printed sheet to simplify your work.

Chapter 9: 9.1, 9.2, 9.3 – Periodic Functions and Fourier Series

Problem Xc9.0-1. (Trigonometric Identities and Integrals)

(a) Use the trigonometric identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$ to derive the trigonometric identity

$$\cos mx \cos nx = \frac{1}{2}\left(\cos((m+n)x) + \cos((m-n)x)\right).$$

(b) Show the details for integrating $\cos mx \cos nx$ for nonnegative integers $m \neq n$ over $-\pi \leq x \leq \pi$.

(c) Derive the trigonometric identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ from the trigonometric identities $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\cos^2\theta + \sin^2\theta = 1$.

(d) Integrate $\cos^2 nx$ for integers $n = 0, 1, 2, 3, \ldots$ over $-\pi \leq x \leq \pi$. Explain geometrically why there are two different answers.

Problem Xc9.0-2. (Orthogonality)

Two vectors $\vec{A}$, $\vec{B}$ are said to be **orthogonal** if their dot product is zero. For vectors of dimension n, this means $a_1b_1 + a_2b_2 + \cdots + a_nb_n = 0$.

(a) The equation $\int_0^1 f(x)g(x)dx = 0$ can be viewed as *the Riemann sum is approximately zero*. Argue that this means $\vec{A}\cdot\vec{B} = 0$ to so many decimal places, where $\vec{A}$ and $\vec{B}$ are **vector samples** of f and g represented in the Riemann sum $h\sum_{j=1}^{n} f(jh)g(jh)$, $h = \frac{1}{n}$.

(b) Prove the orthogonality relation below from the standard one for the trigonometric system on $-\pi \leq x \leq \pi$, by a change of variables. The method leads to six orthogonality relations for the trigonometric system $\{\cos(m\pi x/T)\}_{m=0}^{\infty}$, $\{\sin(n\pi x/T)\}_{n=1}^{\infty}$ on $-T \leq u \leq T$.

$$\int_{-T}^{T} \cos(m\pi u/T)\cos(n\pi u/T)du = \begin{cases} 0 & m \neq n, \\ T & m = n. \end{cases}$$

Problem Xc9.1-1. (Periodic Functions)

(a) Find the period, amplitude and frequency of $\sin^2(4x)$.

(b) Let f be periodic of period 1 and on $0 \leq x \leq 1$ $f(x) = f_0(x)$, where $f_0(x) = 1$ on $0 \leq x < 1/2$, $f_0(x) = 0$ on $1/2 \leq x < 1$, $f_0(1) = 0$. Graph f on $-2 \leq x \leq 3$.

Problem Xc9.1-8. (Sums of Periodic Functions)

(a) Find the period of $\cos x + \cos 3x$.

(b) Find the period of $e^{2\cos 2x}$.

Problem Xc9.1-8. (Periodic Functions)

Explain why $\cos x + \cos 3\pi x$ is not periodic.

Problem Notes. One explanation uses independence of functions. Another explanation analyzes the number of solutions of the equation $\cos x + \cos 2\pi x = 2$. Expected in this case is a graphic and a mathematical argument [use $-2 \le f(x) \le 2$].

Problem Xc9.1-18. (Change of Variables)

(a) Prove that $f(x)$ continuous and T-periodic implies $\displaystyle\int_0^T f(x)dx = \int_{nT}^{nT+T} f(u)du.$

(b) Assume $f(x)$ is 2π-periodic and continuous. Prove that $F(x) = \int_0^x f(u)du$ is 2π-periodic if and only if $\int_0^{2\pi} f(x)dx = 0$.

Problem Xc9.1-20. (Floor Function)

The greatest integer function or **staircase function** is represented using a library function **floor**(x), available in most computer mathematical workbenches, including `MATLAB` and `maple`. Don't confuse **floor** with **trunc** – they are different functions!

(a) Plot **floor**(x) and $x -$ **floor**(x) in `MATLAB` or `maple` on $-3 \le x \le 5$. Programs `Excel` and `OpenOffice` can also be used to make the plot. The **floor** function in `Excel` is $x \to \mathrm{FLOOR}(x, 1)$.

(b) Argue from the graphic that $x -$ **floor**(x) is periodic of period 1.

(c) Define $g(x, T) = x - T\,\mathbf{floor}((x + T/2)/T)$. Show mathematically that $g(x) = x$ on $|x| < T/2$. That the **triangular wave** g is T-periodic can be seen from its graphic.

(d) Plot the 2-periodic **triangular wave** defined by $f(x) = |x - 2\,\mathbf{floor}((x + 1)/2)|$.

Problem Notes: The function f in (d) equals $|g(x, T)|$, where $T = 2$ and g is defined in (c) above. The function $a|g(x, T)|$ is called a **triangular wave** of height a and period T. It is the composition of $u \to a|u|$ and $x \to g(x, T)$.

Problem Xc9.2-5. (Fourier Series Partial Sum Plots)

(a) Plot on $-2\pi \le x \le 3\pi$ the partial sums $s_2(x), s_6(x), s_{12}(x), s_{20}(x)$ of the Fourier series for the sawtooth wave f constructed from $f_1(x) = |x|$ on $|x| \le \pi$:

$$s_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{N} \frac{1}{(2m+1)^2}\cos(2mx + x).$$

The four graphics show the convergence of the partial sums to the limiting Fourier series. This is an example of a *filmstrip* of 4 graphics.

(b) Explain what happens in the graphic of (a) at points of discontinuity of f'.

Problem Xc9.2-5a. (Fourier Series Partial Sum Plots)

(a) Plot on $-2\pi \le x \le 3\pi$ the partial sums $s_2(x), s_6(x), s_{12}(x), s_{20}(x)$ of the Fourier series for the sawtooth wave f constructed from $f_1(x) = (\pi - x)/2$ on $0 < x \le 2\pi$:

$$s_N(x) = \sum_{n=1}^{N} \frac{1}{n}\sin(nx).$$

The four snapshots show the convergence of the partial sums to the limiting Fourier series.

(b) Explain what happens in the graphic of (a) at points of discontinuity of f.

(c) Illustrate Gibb's phenomenon. In particular, graph $|f(x) - s_N(x)|$ on $-2\pi \le x \le 3\pi$, for $N = 6, 12, 20$. Then estimate the jump at points of discontinuity of f'.

Answer: About 1.25.

Problem Xc9.2-8. (Fourier Series Computation and Graphics)

Let f be the 2π-periodic extension of the rectified cosine wave $f_0(x) = |\cos x|$ on $|x| \le \pi$.

(a) Draw a graphic of $f(x)$ on $-4\pi \le x \le 5\pi$.

(b) Show the derivation details for the Fourier series $\displaystyle\frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{4m^2 - 1}\cos 2mx.$

(c) Plot the Fourier series on $-2\pi \le x \le 2\pi$. Explain why it differs from the plot of $f(x)$ on the same interval.

Problem Xc9.2-15. (Fourier Series Computation)

Show the derivation details for the Fourier coefficients of $f(x)$ constructed from $f_2(x) = e^{-|x|}$ on $|x| \le \pi$. The Fourier series is

$$\frac{e^\pi - 1}{\pi e^\pi} + \frac{2}{\pi e^\pi} \sum_{m=1}^{\infty} \frac{e^\pi + (-1)^{m+1}}{m^2+1} \cos mx.$$

Problem Xc9.0-3. (Even and Odd Functions)

(a) Define *even function* and *odd function*. Such functions don't have to be continuous, but they must be defined for all x.

(b) Show the mathematical details in the derivation of the result *(Even)(Odd)=Odd.*

(c) Prove by a u-substitution that $\int_{-p}^{p} f(x)dx = 2\int_0^p f(x)dx$ for an even continuous function f and $\int_{-p}^{p} g(x)dx = 0$ for an odd continuous function $g(x)$.

Problem Xc9.3-7. (Fourier Series Arbitrary Period)

(a) Define f to be the periodic extension of period 4 of the base function $f_0(x) = 1 - x$ on $0 \le x \le 2$, $f_0(x) = -1 - x$ on $-2 \le x \le 0$. Plot $f(x)$ on $-8 \le x \le 6$.

(b) Show the derivation details for the Fourier series of $f(x)$:

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{2n}.$$

Problem Xc9.3-32. (Failure of Term-by-Term Differentiation)

Show that the Fourier series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ of the sawtooth wave f cannot be differentiated term-by-term to obtain the Fourier series of f'.

Problem Xc9.3-34. (Term-by-Term Integration)

Integrate the Fourier series of the triangular wave f constructed from $f_0(x) = x$ on $|x| \le 1$, in order to find the Fourier series of the parabolic wave g constructed from $g_0(x) = x^2$ on $|x| \le 1$.

Chapter 9: 9.3, 9.4 – Fourier Series Methods

Problem Xc9.0-4. (Periodic Extensions)

Lemma 1. The function $\mathbf{tw}(x) = x - \mathbf{floor}(x + 1/2)$ is a **triangular wave** of period 1 with shape x on $|x| < 1/2$.

Lemma 2. Given $f_0(x)$ defined on $|x| \le T/2$, then $f(x) = f_0(T\,\mathbf{tw}(x/T))$ is the T-periodic extension of $f_0(x)$ from $|x| \le T/2$ to $-\infty < x < \infty$.

Assume Lemmas 1 and 2 for this problem.

(a) Plot $f_1(x) = 3\,\mathbf{tw}(x/3)$ on $-6 \le x \le 6$. Document its period on the graphic.

(b) Define $f_2(x) = |\cos(0.5\pi\,\mathbf{tw}(2x/\pi))|$. Make a plot on $-2\pi \le x \le 3\pi$. Document its period on the graphic.

Problem Xc9.0-5. (Even and Odd Periodic Extensions)

Definition. Define $\mathbf{signum}(x) = \begin{cases} \frac{x}{|x|} & x \ne 0, \\ 0 & x = 0. \end{cases}$.

There is no agreement in literature how to define $\mathbf{signum}(0)$. Here, $\mathbf{signum}(x)$ takes on only the values 1, -1 and 0.

(a) Let $p = 2$ and define $g_1(x) = x^2$ on $0 \le x \le p$. Let $g_2(x) = \mathbf{signum}(x) g_1(|x|)$ be the odd extension of g_1 to $|x| \le p$. Let $T = 2p$. Define $f_3(x) = g_2(T\,\mathbf{tw}(x/T))$ to be the odd extension of $g_2(x)$ from $|x| \le p$ to $-\infty < x < \infty$. Plot f_3 on $|x| \le 5$. This sequence of formulas works in general, for any p and any g_1 (no justification requested).

(b) Let $p = 2$ and define $h_1(x) = x^2$ on $0 \le x \le p$. Let $h_2(x) = h_1(|x|)$ be the even extension of h_1 to $|x| \le p$. Let $T = 2p$. Define $h_3(x) = h_2(T\,\mathbf{tw}(x/T))$ to be the even extension of $h_2(x)$ from $|x| \le p$ to $-\infty < x < \infty$. Plot h_3 on $|x| \le 5$. This sequence of formulas works in general, for any p and any h_1 (no justification requested).

Problem Xc9.0-6. **(Dirichlet Kernel Identity)**

Establish by trigonometric identity methods the formula [the right side is called **Dirichlet's Kernel**]

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(nx + \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

Hint: Cross multiply by $2\sin(x/2)$. Expand terms using a trigonometric identity, which produces a telescoping sum.

Problem Xc2.4-7. **(Half-Range Expansions)**

(a) Find a simple algebraic formula for the even π-periodic extension of $f_0(x) = \cos x$ on $0 \le x \le \pi/2$.

(b) Find the Fourier coefficients for the half-range expansion of $f_0(x) = \cos x$ on $0 \le x \le \pi/2$.

Problem Xc9.4-15. **(Half-Range Sine Expansion)**

Find the Fourier coefficients for the half-range sine series expansion of e^x on $0 \le x \le 1$.

Problem Xc9.4-6. **(Complex Fourier Series)**

Find the complex form of the Fourier series for $\sin 3x$ without evaluating any trigonometric integrals.

Hint: Use $\sin u = \frac{1}{2i}\left(e^{iu} - e^{-iu}\right)$.

Problem Xc9.4-11. **(Series Identities)**

Let $x = 0$ in the complex Fourier series expansion of e^x in order to prove the formula

$$\frac{2\pi}{e^{\pi} - e^{-\pi}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + 1}.$$

Chapter 9: 9.5 – One Dimensional Heat Equation

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Problem Xc9.6-13. **(Nonhomogeneous Heat Equation)**

Consider the one-dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx},\ 0 \le x \le \pi,\ t > 0, \\ u(0,t) &= 100, \\ u(\pi,t) &= 50, \\ u(x,0) &= f(x). \end{aligned}$$

Assume $f(x) = 33x$ on $0 < x \le \pi/2$, $f(x) = 33\pi - 33x$ on $\pi/2 < x < \pi$. Find a solution formula for the temperature $u(x,t)$.

Problem Xc9.6-3. **(Heat Conduction in an Insulated Bar)**

Consider the one-dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx},\ 0 \le x \le 1,\ t > 0, \\ u_x(0,t) &= 0, \\ u_x(1,t) &= 0, \\ u(x,0) &= \cos \pi x \end{aligned}$$

Find a solution formula for the temperature $u(x,t)$ at location x along the bar at time t. Hint: Don't integrate!

Remark. Asmar's matching problem 3.6-3 has a piecewise example, using $u(x,0) = f(x)$. See the maple advice for problem 9.5-13, to handle that case.

Chapter 9: 9.6 – One Dimensional Wave Equation

Problem Xc9.6-1. (Wave Equation)

Derive the equation $u_{tt} = 10^5 u_{xx}$ for the vibrations of a stretched homogeneous string with linear density $\rho = 0.001$ kg/m and tension $\tau = 100$ N, with no forces other than the tension. State all assumptions used to obtain the model. Make the presentation brief, by referencing a textbook for derivation details and results.

Problem Xc9.6-9a. (Separation of Variables)

Solve $u_{tt} = u_{xx}$, $u(0,t) = u(1,t) = 0$, $u(x,0) = x(1-x)$, $u_t(x,0) = \sin \pi x$, $t \geq 0$, $0 \leq x \leq 1$. The model is for a guitar string of unit length.

Problem Xc9.6-9b. (Filmstrip Plots)

Plot partial sums of the answer to the previous problem,

$$u(x,t) = \frac{1}{\pi}\sin(\pi x)\sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3}\sin(2m\pi x + \pi x)\cos(2m\pi t + \pi t),$$

at $t = 0, 1, 2, 3$. Choose the number of series terms for the four graphics by making the first graphic match $x(1-x)$ on $0 \leq x \leq 1$. This filmstrip has 4 frames, each frame corresponding to a time t. A frame has graph window $0 \leq x \leq 1$, $a \leq u \leq b$ (you must choose a, b).

Problem Xc9.6-9c. (Surface Plot)

Plot a specific partial sum of the answer

$$u(x,t) = \frac{1}{\pi}\sin(\pi x)\sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3}\sin(2m\pi x + \pi x)\cos(2m\pi t + \pi t)$$

on the domain $0 \leq x \leq 1$, $0 \leq t \leq 4$. Use all features possible of the 3D graphics program in order to produce the best plot with fine accuracy, view and colors.

Problem Xc9.6-13. (Damped Vibrations of a String)

Solve the problem

$$\begin{aligned} u_{tt}(x,t) + u_t(x,t) &= u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(\pi,t) &= 0, \\ u(x,0) &= \sin x, \\ u_t(x,0) &= 0. \end{aligned}$$

Problem Xc9.6-1. (Vibrating Finite String)

Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ with boundary and initial conditions $u(0,t) = u(L,t) = 0$, $u(x,0) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L} + \frac{2}{5}\sin\frac{7\pi x}{L}$, $u_t(x,0) = 0$, $0 < x < L$, $t \geq 0$. Use the series formula $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$.

References: Edwards-Penney section 9.6 (2280 textbook) and Asmar's text, *PDE and BVP*, section 1.2.

Problem Xc9.6-2. (Loudness)

The fraction of the loudness associated with the fundamental tone (b_1-term in the series) is the quotient

$$F_1 = \frac{\left(n^2 b_n^2\right)\big|_{n=1}}{\sum_{k=1}^{\infty} k^2 b_k^2}$$

Find an approximation to the percentage $100F_1$.

References: ProbXc9.6-1. The discussion of music in E&P includes a derivation of the formula for the percentage loudness $100F_n$.

Chapter 9: 9.6 – d'Alembert's Method

Problem Xc9.6-15. **(d'Alembert's Solution)**

Consider the problem

$$\begin{aligned} u_{tt} &= u_{xx}, \ 0 \le x \le 1, \, t \ge 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= 0. \end{aligned}$$

Assume $f(x) = 4x$ on $0 \le x \le 0.25$, $f(x) = 2 - 4x$ on $0.25 < x \le 0.5$, $f(x) = 0$ on $0.5 < x \le 1$.

(a) Find a solution formula for $u(x,t)$ using d'Alembert's method.

(b) Plot a 3-frame filmstrip of the string shape at times $t = 0, 0.25, 0.5$.

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# EXAMPLE. Let f(x)=4x on [0,.25], f(x)=2-4x on [.25,.5], f(x)=0 otherwise
# Asmar 3.4-15, D'Alembert's solution of the wave equation, f=pulses,g=0
pulse:=(x,a,b)->piecewise(x<a,0,x<b,1,0);
f:=x->4*x*pulse(x,0,1/4)+(2-4*x)*pulse(x,1/4,1/2);
#plot(f(x),x=0..1);
F:=x->piecewise(x<0,-f(-x),f(x)); # Odd extension of f(x)
plot(F(x),x=-1..1);
u:=(x,t)->(1/2)*(F(x+t)+F(x-t));
#plot(u(x,0.7),x=-2..2);
plots[animate]( plot, [u(x,t),x=-3..3], t=0..1.5, trace=0, frames=50 );
```

Problem Xc9.6-18. **(Energy Conservation and d'Alembert's Solution)**

Define

$$E(t) = \frac{1}{2} \int_0^L \left(u_t^2(x,t) + c^2 u_x^2(x,t) \right) dx.$$

Prove the energy conservation law, which says that the energy during free vibrations of a string is constant for all time.

Problem Notes. Show $dE/dt = 0$.

End of extra credit problems chapter 9.