Matrix Exponential: Putzer Formula for e^{At}

Variation of Parameters for Linear Dynamical Systems

Undetermined Coefficients for Linear Dynamical Systems

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The 2×2 Matrix Exponential e^{At}

Definition. The **matrix exponential** e^{At} is the $n \times n$ matrix $\Phi(t)$ defined by

$$(1)\quad \frac{d}{dt}\Phi = A\Phi, \qquad (2)\quad \Phi(0) = I.$$

Alternatively, Φ is the augmented matrix of solution vectors for the n problems $\frac{d}{dt}\vec{v}_k = A\vec{v}_k$, $\vec{v}_k(0) =$ column k of I, $1 \le k \le n$.

Example. A 2×2 matrix A has exponential matrix e^{At} with columns equal to the solutions of the two problems

$$\begin{cases} \frac{d}{dt}\vec{\mathbf{v}}_1(t) = A\vec{\mathbf{v}}_1(t), \\ \vec{\mathbf{v}}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \qquad \begin{cases} \frac{d}{dt}\vec{\mathbf{v}}_2(t) = A\vec{\mathbf{v}}_2(t), \\ \vec{\mathbf{v}}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

Briefly, the 2×2 matrix $\Phi(t) = e^{At}$ satisfies the two conditions

$$(1)\quad \frac{d}{dt}\Phi(t) = A\Phi(t), \quad (2)\quad \Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Putzer Matrix Exponential Formula for 2×2 Matrices

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$$

A is 2×2, $\lambda_1 \neq \lambda_2$ real.

$$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t}(A - \lambda_1 I)$$

A is 2×2, $\lambda_1 = \lambda_2$ real.

$$e^{At} = e^{at} \cos bt\, I + \frac{e^{at} \sin bt}{b}(A - aI)$$

A is 2×2, $\lambda_1 = \overline{\lambda}_2 = a + ib$, $b > 0$.

How to Remember Putzer's 2 × 2 Formula

The expressions

$$
\begin{aligned}
e^{At} &= r_1(t)I + r_2(t)(A - \lambda_1 I), \\
r_1(t) &= e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}
\end{aligned} \tag{1}
$$

are enough to generate all three formulas. Fraction r_2 is the $d/d\lambda$-Newton difference quotient for r_1. Then r_2 limits as $\lambda_2 \to \lambda_1$ to the $d/d\lambda$-derivative $te^{\lambda_1 t}$. Therefore, the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$
r_2(t) = \frac{z - \overline{z}}{w - \overline{w}} = \frac{\sin bt}{b}.
$$

Taking real parts of expression (1) gives the complex case formula.

Variation of Parameters

Theorem 1 (Variation of Parameters for Linear Systems)
Let A be a constant $n \times n$ matrix and $\mathbf{F}(t)$ a continuous function near $t = t_0$. The unique solution $\mathbf{x}(t)$ of the matrix initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{F}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

is given by the **variation of parameters formula**

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_{t_0}^{t} e^{-rA}\mathbf{F}(r)dr. \tag{2}$$

Theorem 2 (Polynomial Solutions)
Let $\boldsymbol{f}(\boldsymbol{t})$ be a polynomial of degree $\boldsymbol{k}$. Assume $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ constant invertible matrix. Then $\mathbf{u}' = \boldsymbol{A}\mathbf{u} + \boldsymbol{f}(\boldsymbol{t})\mathbf{c}$ has a polynomial solution $\mathbf{u}(\boldsymbol{t}) = \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!}$ of degree $\boldsymbol{k}$ with vector coefficients $\{\mathbf{c}_j\}$ given by the relations

$$\mathbf{c}_j = -\sum_{i=j}^{k} f^{(i)}(0) A^{j-i-1}\mathbf{c}, \quad 0 \le j \le k.$$

Changes from nth Order Undetermined Coefficients. The nth order theory using Rule I and Rule II is replaced by

> **Systems Rule for Undetermined Coefficients.** Assume $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}} + \vec{F}(t)$. Extract all Euler atoms from $\vec{F}, \vec{F}', \ldots$. Don't replace atoms by groups (Rule II). Instead, extend each existing group (Rule I) by adding $m - 1$ higher power terms x^k(base atom) to the group, where m is the multiplicity of the root for the base atom in the characteristic equation $|A - rI| = 0$. The trial solution is a linear combination of the final atom list with vector coefficients.