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# Chapter 11

## Systems of Differential Equations

### Contents

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11.1	Examples of Systems . . . . .	680
11.2	Basic First-order System Methods . . . . .	704
11.3	Structure of Linear Systems . . . . .	712
11.4	Matrix Exponential . . . . .	720
11.5	The Eigenanalysis Method . . . . .	728
11.6	Jordan Form and Eigenanalysis . . . . .	735
11.7	Nonhomogeneous Linear Systems . . . . .	749
11.8	Second-order Systems . . . . .	752
11.9	Numerical Methods for Systems . . . . .	755

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**Linear systems.** A **linear system** is a system of differential equations of the form

$$(1) \quad \begin{array}{rcl} x_1' & = & a_{11}x_1 + \cdots + a_{1n}x_n + f_1, \\ x_2' & = & a_{21}x_1 + \cdots + a_{2n}x_n + f_2, \\ & \vdots & \vdots \quad \cdots \quad \vdots \quad \quad \quad \vdots \\ x_m' & = & a_{m1}x_1 + \cdots + a_{mn}x_n + f_m, \end{array}$$

where  $' = d/dt$ . Given are the functions  $a_{ij}(t)$  and  $f_j(t)$  on some interval  $a < t < b$ . The unknowns are the functions  $x_1(t), \dots, x_n(t)$ .

The system is called **homogeneous** if all  $f_j = 0$ , otherwise it is called **non-homogeneous**.

**Matrix Notation for Systems.** A non-homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t),$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

## 11.1 Examples of Systems

Brine Tank Cascade	680
Cascades and Compartment Analysis	681
Recycled Brine Tank Cascade	682
Pond Pollution	683
Home Heating	685
Chemostats and Microorganism Culturing	687
Irregular Heartbeats and Lidocaine	689
Nutrient Flow in an Aquarium	690
Biomass Transfer	691
Pesticides in Soil and Trees	692
Forecasting Prices	693
Coupled Spring-Mass Systems	694
Boxcars	696
Electrical Network I	697
Electrical Network II	698
Logging Timber by Helicopter	699
Earthquake Effects on Buildings	700

### Brine Tank Cascade

Let brine tanks  $A$ ,  $B$ ,  $C$  be given of volumes 20, 40, 60, respectively, as in Figure 1.

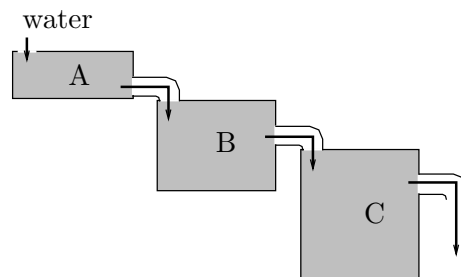


Figure 1. Three brine tanks in cascade.

It is supposed that fluid enters tank  $A$  at rate  $r$ , drains from  $A$  to  $B$  at rate  $r$ , drains from  $B$  to  $C$  at rate  $r$ , then drains from tank  $C$  at rate  $r$ . Hence the volumes of the tanks remain constant. Let  $r = 10$ , to illustrate the ideas.

Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.

Let  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  denote the amount of salt at time  $t$  in each tank. We suppose added to tank  $A$  **water containing no salt**. Therefore, the salt in all the tanks is eventually lost from the drains. The cascade is modeled by the **chemical balance law**

$$\text{rate of change} = \text{input rate} - \text{output rate}.$$

Application of the balance law, justified below in *compartment analysis*, results in the triangular differential system

$$\begin{aligned}x_1' &= -\frac{1}{2}x_1, \\x_2' &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\x_3' &= \frac{1}{4}x_2 - \frac{1}{6}x_3.\end{aligned}$$

The solution, to be justified later in this chapter, is given by the equations

$$\begin{aligned}x_1(t) &= x_1(0)e^{-t/2}, \\x_2(t) &= -2x_1(0)e^{-t/2} + (x_2(0) + 2x_1(0))e^{-t/4}, \\x_3(t) &= \frac{3}{2}x_1(0)e^{-t/2} - 3(x_2(0) + 2x_1(0))e^{-t/4} \\&\quad + (x_3(0) - \frac{3}{2}x_1(0) + 3(x_2(0) + 2x_1(0)))e^{-t/6}.\end{aligned}$$

## Cascades and Compartment Analysis

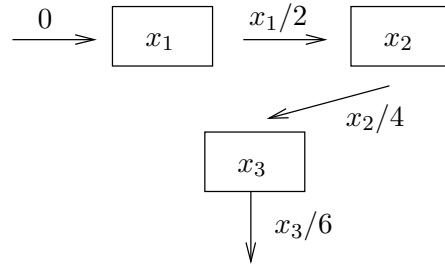
A **linear cascade** is a diagram of **compartments** in which input and output rates have been assigned from one or more different compartments. The diagram is a succinct way to summarize and document the various rates.

The method of **compartment analysis** translates the diagram into a system of linear differential equations. The method has been used to derive applied models in diverse topics like ecology, chemistry, heating and cooling, kinetics, mechanics and electricity.

**The method.** Refer to Figure 2. A compartment diagram consists of the following components.

Variable Names      Each **compartment** is labelled with a variable  $X$ .

Arrows	Each arrow is labelled with a <b>flow rate</b> $R$ .
Input Rate	An arrowhead pointing at compartment $X$ documents <b>input rate</b> $R$ .
Output Rate	An arrowhead pointing away from compartment $X$ documents <b>output rate</b> $R$ .



**Figure 2. Compartment analysis diagram.**

The diagram represents the classical brine tank problem of Figure 1.

Assembly of the single linear differential equation for a diagram compartment  $X$  is done by writing  $dX/dt$  for the left side of the differential equation and then algebraically adding the input and output rates to obtain the right side of the differential equation, according to the **balance law**

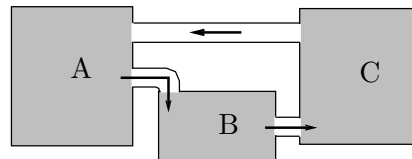
$$\frac{dX}{dt} = \text{sum of input rates} - \text{sum of output rates}$$

By convention, a compartment with no arriving arrowhead has input zero, and a compartment with no exiting arrowhead has output zero. Applying the balance law to Figure 2 gives one differential equation for each of the three compartments  $x_1$ ,  $x_2$ ,  $x_3$ .

$$\begin{aligned}x_1' &= 0 - \frac{1}{2}x_1, \\x_2' &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\x_3' &= \frac{1}{4}x_2 - \frac{1}{6}x_3.\end{aligned}$$

## Recycled Brine Tank Cascade

Let brine tanks  $A$ ,  $B$ ,  $C$  be given of volumes 60, 30, 60, respectively, as in Figure 3.



**Figure 3. Three brine tanks in cascade with recycling.**

Suppose that fluid drains from tank  $A$  to  $B$  at rate  $r$ , drains from tank  $B$  to  $C$  at rate  $r$ , then drains from tank  $C$  to  $A$  at rate  $r$ . The tank

volumes remain constant due to constant recycling of fluid. For purposes of illustration, let  $r = 10$ .

Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.

Let  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  denote the amount of salt at time  $t$  in each tank. No salt is lost from the system, due to recycling. Using compartment analysis, the recycled cascade is modeled by the non-triangular system

$$\begin{aligned}x_1' &= -\frac{1}{6}x_1 && + \frac{1}{6}x_3, \\x_2' &= \frac{1}{6}x_1 && - \frac{1}{3}x_2, \\x_3' &= && \frac{1}{3}x_2 - \frac{1}{6}x_3.\end{aligned}$$

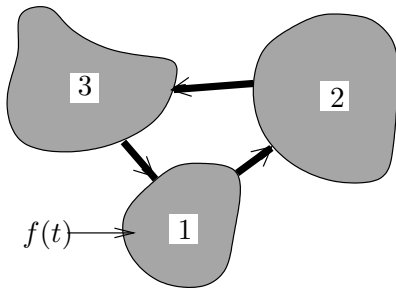
The solution is given by the equations

$$\begin{aligned}x_1(t) &= c_1 + (c_2 - 2c_3)e^{-t/3} \cos(t/6) + (2c_2 + c_3)e^{-t/3} \sin(t/6), \\x_2(t) &= \frac{1}{2}c_1 + (-2c_2 - c_3)e^{-t/3} \cos(t/6) + (c_2 - 2c_3)e^{-t/3} \sin(t/6), \\x_3(t) &= c_1 + (c_2 + 3c_3)e^{-t/3} \cos(t/6) + (-3c_2 + c_3)e^{-t/3} \sin(t/6).\end{aligned}$$

At infinity,  $x_1 = x_3 = c_1$ ,  $x_2 = c_1/2$ . The meaning is that the total amount of salt is uniformly distributed in the tanks, in the ratio 2 : 1 : 2.

## Pond Pollution

Consider three ponds connected by streams, as in Figure 4. The first pond has a pollution source, which spreads via the connecting streams to the other ponds. The plan is to determine the amount of pollutant in each pond.



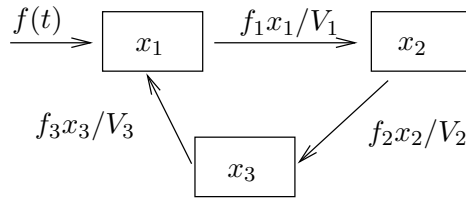
**Figure 4.** Three ponds 1, 2, 3 of volumes  $V_1$ ,  $V_2$ ,  $V_3$  connected by streams. The pollution source  $f(t)$  is in pond 1.

Assume the following.

- Symbol  $f(t)$  is the pollutant flow rate into pond 1 (lb/min).
- Symbols  $f_1$ ,  $f_2$ ,  $f_3$  denote the pollutant flow rates out of ponds 1, 2, 3, respectively (gal/min). It is assumed that the pollutant is well-mixed in each pond.

- The three ponds have volumes  $V_1, V_2, V_3$  (gal), which remain constant.
- Symbols  $x_1(t), x_2(t), x_3(t)$  denote the amount (lbs) of pollutant in ponds 1, 2, 3, respectively.

The pollutant flux is the flow rate times the pollutant concentration, e.g., pond 1 is emptied with flux  $f_1$  times  $x_1(t)/V_1$ . A compartment analysis is summarized in the following diagram.



**Figure 5. Pond diagram.**  
The compartment diagram represents the three-pond pollution problem of Figure 4.

The diagram plus compartment analysis gives the following differential equations.

$$\begin{aligned}x_1'(t) &= \frac{f_3}{V_3} x_3(t) - \frac{f_1}{V_1} x_1(t) + f(t), \\x_2'(t) &= \frac{f_1}{V_1} x_1(t) - \frac{f_2}{V_2} x_2(t), \\x_3'(t) &= \frac{f_2}{V_2} x_2(t) - \frac{f_3}{V_3} x_3(t).\end{aligned}$$

For a specific numerical example, take  $f_i/V_i = 0.001$ ,  $1 \leq i \leq 3$ , and let  $f(t) = 0.125$  lb/min for the first 48 hours (2880 minutes), thereafter  $f(t) = 0$ . We expect due to uniform mixing that after a long time there will be  $(0.125)(2880) = 360$  pounds of pollutant uniformly deposited, which is 120 pounds per pond.

Initially,  $x_1(0) = x_2(0) = x_3(0) = 0$ , if the ponds were pristine. The specialized problem for the first 48 hours is

$$\begin{aligned}x_1'(t) &= 0.001 x_3(t) - 0.001 x_1(t) + 0.125, \\x_2'(t) &= 0.001 x_1(t) - 0.001 x_2(t), \\x_3'(t) &= 0.001 x_2(t) - 0.001 x_3(t), \\x_1(0) &= x_2(0) = x_3(0) = 0.\end{aligned}$$

The solution to this system is

$$\begin{aligned}x_1(t) &= e^{-\frac{3t}{2000}} \left( \frac{125\sqrt{3}}{9} \sin\left(\frac{\sqrt{3}t}{2000}\right) - \frac{125}{3} \cos\left(\frac{\sqrt{3}t}{2000}\right) \right) + \frac{125}{3} + \frac{t}{24}, \\x_2(t) &= -\frac{250\sqrt{3}}{9} e^{-\frac{3t}{2000}} \sin\left(\frac{\sqrt{3}t}{2000}\right) + \frac{t}{24}, \\x_3(t) &= e^{-\frac{3t}{2000}} \left( \frac{125}{3} \cos\left(\frac{\sqrt{3}t}{2000}\right) + \frac{125\sqrt{3}}{9} \sin\left(\frac{\sqrt{3}t}{2000}\right) \right) + \frac{t}{24} - \frac{125}{3}.\end{aligned}$$

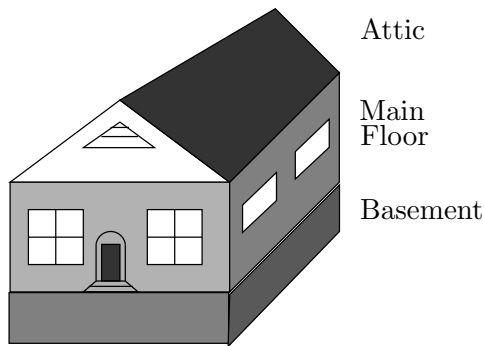
After 48 hours elapse, the approximate pollutant amounts in pounds are

$$x_1(2880) = 162.30, \quad x_2(2880) = 119.61, \quad x_3(2880) = 78.08.$$

It should be remarked that the system above is altered by replacing 0.125 by zero, in order to predict the state of the ponds after 48 hours. The corresponding homogeneous system has an equilibrium solution  $x_1(t) = x_2(t) = x_3(t) = 120$ . This constant solution is the limit at infinity of the solution to the homogeneous system, using the initial values  $x_1(0) \approx 162.30$ ,  $x_2(0) \approx 119.61$ ,  $x_3(0) \approx 78.08$ .

## Home Heating

Consider a typical home with attic, basement and insulated main floor.



**Figure 6. Typical home with attic and basement.**

The below-grade basement and the attic are un-insulated. Only the main living area is insulated.

It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation. The walls and floor in the basement are insulated by earth. The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement. We will analyze the changing temperatures in the three levels using Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

**Initial data.** Assume it is winter time and the outside temperature is constantly  $35^\circ\text{F}$  during the day. Also assumed is a basement earth temperature of  $45^\circ\text{F}$ . Initially, the heat is off for several days. The initial values at noon ( $t = 0$ ) are then  $x(0) = 45$ ,  $y(0) = z(0) = 35$ .

**Portable heater.** A small electric heater is turned on at noon, with thermostat set for  $100^\circ\text{F}$ . When the heater is running, it provides a  $20^\circ\text{F}$  rise per hour, therefore it takes some time to reach  $100^\circ\text{F}$  (probably never!). Newton's cooling law

Temperature rate =  $k$ (Temperature difference)

will be applied to five boundary surfaces: (0) the basement walls and floor, (1) the basement ceiling, (2) the main floor walls, (3) the main floor ceiling, and (4) the attic walls and ceiling. Newton's cooling law gives positive cooling constants  $k_0, k_1, k_2, k_3, k_4$  and the equations

$$\begin{aligned}x' &= k_0(45 - x) + k_1(y - x), \\y' &= k_1(x - y) + k_2(35 - y) + k_3(z - y) + 20, \\z' &= k_3(y - z) + k_4(35 - z).\end{aligned}$$

The insulation constants will be defined as  $k_0 = 1/2, k_1 = 1/2, k_2 = 1/4, k_3 = 1/4, k_4 = 3/4$  to reflect insulation quality. The reciprocal  $1/k$  is approximately the amount of time in hours required for 63% of the temperature difference to be exchanged. For instance, 4 hours elapse for the main floor. The model:

$$\begin{aligned}x' &= \frac{1}{2}(45 - x) + \frac{1}{2}(y - x), \\y' &= \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20, \\z' &= \frac{1}{4}(y - z) + \frac{3}{4}(35 - z).\end{aligned}$$

The homogeneous solution in vector form is given in terms of constants  $a = 1 + \sqrt{5}/4, b = 1 - \sqrt{5}/4$ , and arbitrary constants  $c_1, c_2, c_3$  by the formula

$$\begin{pmatrix} x_h(t) \\ y_h(t) \\ z_h(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + c_2 e^{-at} \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix} + c_3 e^{-bt} \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix}.$$

A particular solution is an equilibrium solution

$$\begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix} = \begin{pmatrix} \frac{620}{11} \\ \frac{745}{11} \\ \frac{475}{11} \end{pmatrix}.$$

The homogeneous solution has limit zero at infinity, hence the temperatures of the three spaces hover around  $x = 56.4, y = 67.7, z = 43.2$  degrees Fahrenheit. Specific information can be gathered by solving for  $c_1, c_2, c_3$  according to the initial data  $x(0) = 45, y(0) = z(0) = 35$ . The answers are

$$c_1 = 5, \quad c_2 = \frac{25}{2} + \frac{7}{2}\sqrt{5}, \quad c_3 = \frac{25}{2} - \frac{7}{2}\sqrt{5}.$$

**Underpowered heater.** To the main floor each hour is added 20°F, but the heat escapes at a substantial rate, so that after one hour  $y \approx 68^\circ\text{F}$ .



After five hours,  $y \approx 68^\circ\text{F}$ . The heater in this example is so inadequate that even after many hours, the main living area is still under  $69^\circ\text{F}$ .

**Forced air furnace.** Replacing the space heater by a normal furnace adds the difficulty of **switches** in the input, namely, the thermostat turns off the furnace when the main floor temperature reaches  $70^\circ\text{F}$ , and it turns it on again after a  $4^\circ\text{F}$  temperature drop. We will suppose that the furnace has four times the BTU rating of the space heater, which translates to an  $80^\circ\text{F}$  temperature rise per hour. The study of the forced air furnace requires two differential equations, one with 20 replaced by 80 (DE 1, furnace on) and the other with 20 replaced by 0 (DE 2, furnace off). The plan is to use the first differential equation on time interval  $0 \leq t \leq t_1$ , then switch to the second differential equation for time interval  $t_1 \leq t \leq t_2$ . The time intervals are selected so that  $y(t_1) = 70$  (the thermostat setting) and  $y(t_2) = 66$  (thermostat setting less 4 degrees). Numerical work gives the following results.

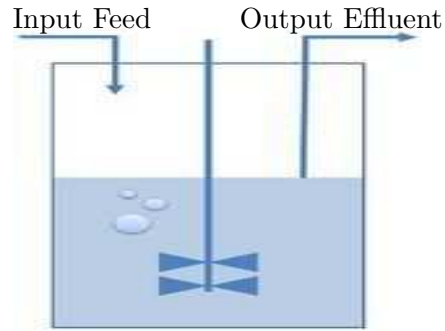
Time in minutes	Main floor temperature	Model	Furnace
31.6	70	DE 1	on
40.9	66	DE 2	off
45.3	70	DE 1	on
54.6	66	DE 2	off

The reason for the non-uniform times between furnace cycles can be seen from the model. Each time the furnace cycles, heat enters the main floor, then escapes through the other two levels. Consequently, the initial conditions applied to models 1 and 2 are changing, resulting in different solutions to the models on each switch.

## Chemostats and Microorganism Culturing

A vessel into which nutrients are pumped, to feed a microorganism, is called a **chemostat**<sup>1</sup>. Uniform distributions of microorganisms and nutrients are assumed, for example, due to stirring effects. The pumping is matched by draining to keep the volume constant.

<sup>1</sup>The October 14, 2004 issue of the journal *Nature* featured a study of the co-evolution of a common type of bacteria, *Escherichia coli*, and a virus that infects it, called bacteriophage T7. Postdoctoral researcher Samantha Forde set up "microbial communities of bacteria and viruses with different nutrient levels in a series of chemostats – glass culture tubes that provide nutrients and oxygen and siphon off wastes."



**Figure 7. A Basic Chemostat.** A stirred bio-reactor operated as a chemostat, with continuous inflow and outflow. The flow rates are controlled to maintain a constant culture volume.

In a typical chemostat, one nutrient is kept in short supply while all others are abundant. We consider here the question of **survival** of the organism subject to the limited resource. The problem is quantified as follows:

$x(t)$  = the concentration of the limited nutrient in the vessel,

$y(t)$  = the concentration of organisms in the vessel.

A special case of the derivation in J.M. Cushing's text for the organism *E. Coli*<sup>2</sup> is the set of **nonlinear** differential equations<sup>3</sup>

$$(2) \quad \begin{aligned} x' &= -0.075x + (0.075)(0.005) - \frac{1}{63}g(x)y, \\ y' &= -0.075y + g(x)y, \end{aligned}$$

where  $g(x) = 0.68x(0.0016 + x)^{-1}$ . Of special interest to the study of this equation are two linearized equations at equilibria, given by

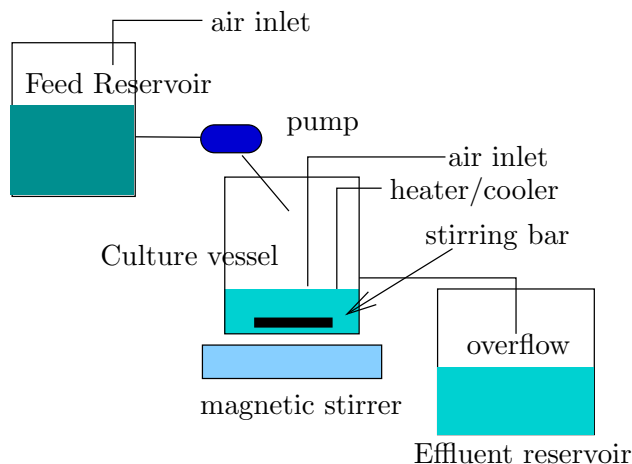
$$(3) \quad \begin{aligned} u_1' &= -0.075 u_1 - 0.008177008175 u_2, \\ u_2' &= 0.4401515152 u_2, \end{aligned}$$

$$(4) \quad \begin{aligned} v_1' &= -1.690372243 v_1 - 0.001190476190 v_2, \\ v_2' &= 101.7684513 v_1. \end{aligned}$$

Although we cannot solve the nonlinear system explicitly, nevertheless there are explicit formulas for  $u_1$ ,  $u_2$ ,  $v_1$ ,  $v_2$  that complete the picture of how solutions  $x(t)$ ,  $y(t)$  behave at  $t = \infty$ . The result of the analysis is that *E. Coli* survives indefinitely in this vessel at concentration  $y \approx 0.3$ .

<sup>2</sup>In a biology Master's thesis, two strains of *Escherichia coli* were grown in a glucose-limited chemostat coupled to a modified Robbins device containing plugs of silicone rubber urinary catheter material. Reference: Jennifer L. Adams and Robert J. C. McLean, *Applied and Environmental Microbiology*, September 1999, p. 4285-4287, Vol. 65, No. 9.

<sup>3</sup>More details can be found in *The Theory of the Chemostat Dynamics of Microbial Competition*, ISBN-13: 9780521067348, by Hal Smith and Paul Waltman, June 2008.



**Figure 8. Laboratory Chemostat.**

The components are the **Feed reservoir**, which contains the nutrients, a stirred chemical reactor labeled the **Culture vessel**, and the **Effluent reservoir**, which holds the effluent overflow from the reactor.

## Irregular Heartbeats and Lidocaine

The human malady of **ventricular arrhythmia** or irregular heartbeat is treated clinically using the drug **lidocaine**.



**Figure 9. Xylocaine label, a brand name for the drug lidocaine.**

To be effective, the drug has to be maintained at a bloodstream concentration of 1.5 milligrams per liter, but concentrations above 6 milligrams per liter are considered lethal in some patients. The actual dosage depends upon body weight. The adult dosage maximum for ventricular tachycardia is reported at 3 mg/kg.<sup>4</sup> The drug is supplied in 0.5%, 1% and 2% solutions, which are stored at room temperature.

A differential equation model for the dynamics of the drug therapy uses

$$\begin{aligned}x(t) &= \text{amount of lidocaine in the bloodstream,} \\y(t) &= \text{amount of lidocaine in body tissue.}\end{aligned}$$

A typical set of equations, valid for a special body weight only, appears below; for more detail see J.M. Cushing's text [?].

$$(5) \quad \begin{aligned}x'(t) &= -0.09x(t) + 0.038y(t), \\y'(t) &= 0.066x(t) - 0.038y(t).\end{aligned}$$

<sup>4</sup>Source: **Family Practice Notebook**, <http://www.fpnotebook.com/>. The author is Scott Moses, MD, who practises in Lino Lakes, Minnesota.

The physically significant initial data is zero drug in the bloodstream  $x(0) = 0$  and injection dosage  $y(0) = y_0$ . The answers:

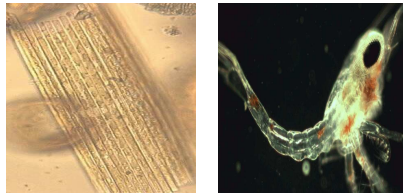
$$\begin{aligned}x(t) &= -0.3367y_0e^{-0.1204t} + 0.3367y_0e^{-0.0076t}, \\y(t) &= 0.2696y_0e^{-0.1204t} + 0.7304y_0e^{-0.0076t}.\end{aligned}$$

The answers can be used to estimate the maximum possible safe dosage  $y_0$  and the duration of time that the drug *lidocaine* is effective.

## Nutrient Flow in an Aquarium

Consider a vessel of water containing a radioactive isotope, to be used as a tracer for the food chain, which consists of aquatic plankton varieties  $A$  and  $B$ .

Plankton are aquatic organisms that drift with the currents, typically in an environment like Chesapeake Bay. Plankton can be divided into two groups, phytoplankton and zooplankton. The phytoplankton are *plant-like* drifters: diatoms and other alga. Zooplankton are *animal-like* drifters: copepods, larvae, and small crustaceans.



**Figure 10. Left: *Bacillaria paxillifera*, phytoplankton. Right: *Anomura Galathea* zoea, zooplankton.**

Let

$x(t)$  = isotope concentration in the water,

$y(t)$  = isotope concentration in  $A$ ,

$z(t)$  = isotope concentration in  $B$ .

Typical differential equations are

$$\begin{aligned}x'(t) &= -3x(t) + 6y(t) + 5z(t), \\y'(t) &= 2x(t) - 12y(t), \\z'(t) &= x(t) + 6y(t) - 5z(t).\end{aligned}$$

The answers are

$$\begin{aligned}x(t) &= 6c_1 + (1 + \sqrt{6})c_2e^{(-10+\sqrt{6})t} + (1 - \sqrt{6})c_3e^{(-10-\sqrt{6})t}, \\y(t) &= c_1 + c_2e^{(-10+\sqrt{6})t} + c_3e^{(-10-\sqrt{6})t}, \\z(t) &= \frac{12}{5}c_1 - (2 + \sqrt{1.5})c_2e^{(-10+\sqrt{6})t} + (-2 + \sqrt{1.5})c_3e^{(-10-\sqrt{6})t}.\end{aligned}$$

The constants  $c_1$ ,  $c_2$ ,  $c_3$  are related to the initial radioactive isotope concentrations  $x(0) = x_0$ ,  $y(0) = 0$ ,  $z(0) = 0$ , by the  $3 \times 3$  system of linear algebraic equations

$$\begin{aligned} 6c_1 + (1 + \sqrt{6})c_2 + (1 - \sqrt{6})c_3 &= x_0, \\ c_1 + c_2 + c_3 &= 0, \\ \frac{12}{5}c_1 - (2 + \sqrt{1.5})c_2 + (-2 + \sqrt{1.5})c_3 &= 0. \end{aligned}$$

## Biomass Transfer

Consider a European forest having one or two varieties of trees. We select some of the oldest trees, those expected to die off in the next few years, then follow the cycle of living trees into dead trees. The dead trees eventually decay and fall from seasonal and biological events. Finally, the fallen trees become humus.



**Figure 11. Forest Biomass.** Total biomass is a parameter used to assess atmospheric carbon that is harvested by trees. Forest management uses biomass subclasses to classify fire risk.

Let variables  $x$ ,  $y$ ,  $z$ ,  $t$  be defined by

$$\begin{aligned} x(t) &= \text{biomass decayed into humus,} \\ y(t) &= \text{biomass of dead trees,} \\ z(t) &= \text{biomass of living trees,} \\ t &= \text{time in decades (decade = 10 years).} \end{aligned}$$

A typical biological model is

$$\begin{aligned} x'(t) &= -x(t) + 3y(t), \\ y'(t) &= -3y(t) + 5z(t), \\ z'(t) &= -5z(t). \end{aligned}$$

Suppose there are no dead trees and no humus at  $t = 0$ , with initially  $z_0$  units of living tree biomass. These assumptions imply initial conditions  $x(0) = y(0) = 0$ ,  $z(0) = z_0$ . The solution is

$$\begin{aligned}x(t) &= \frac{15}{8}z_0 \left( e^{-5t} - 2e^{-3t} + e^{-t} \right), \\y(t) &= \frac{5}{2}z_0 \left( -e^{-5t} + e^{-3t} \right), \\z(t) &= z_0 e^{-5t}.\end{aligned}$$

The live tree biomass  $z(t) = z_0 e^{-5t}$  decreases according to a Malthusian decay law from its initial size  $z_0$ . It decays to 60% of its original biomass in one year. Interesting calculations that can be made from the other formulas include the future dates when the dead tree biomass and the humus biomass are maximum. The predicted dates are approximately 2.5 and 8 years hence, respectively.

The predictions made by this model are trends extrapolated from rate observations in the forest. Like weather prediction, it is a calculated guess that disappoints on a given day and from the outset has no predictable answer.

Total biomass is considered an important parameter to assess atmospheric carbon that is harvested by trees. Biomass estimates for forests since 1980 have been made by satellite remote sensing data with instances of 90% accuracy (*Science* 87(5), September 2004).

## Pesticides in Soil and Trees

A Washington cherry orchard was sprayed with pesticides.



**Figure 12.** Cherries in June.

Assume that a negligible amount of pesticide was sprayed on the soil. Pesticide applied to the trees has a certain outflow rate to the soil, and conversely, pesticide in the soil has a certain uptake rate into the trees. Repeated applications of the pesticide are required to control the insects, which implies the pesticide levels in the trees varies with time. Quantize the pesticide spraying as follows.

$x(t)$  = amount of pesticide in the trees,  
 $y(t)$  = amount of pesticide in the soil,  
 $r(t)$  = amount of pesticide applied to the trees,  
 $t$  = time in years.

A typical model is obtained from input-output analysis, similar to the brine tank models:

$$\begin{aligned}x'(t) &= 2x(t) - y(t) + r(t), \\y'(t) &= 2x(t) - 3y(t).\end{aligned}$$

In a pristine orchard, the initial data is  $x(0) = 0$ ,  $y(0) = 0$ , because the trees and the soil initially harbor no pesticide. The solution of the model obviously depends on  $r(t)$ . The nonhomogeneous dependence is treated by the method of variation of parameters *infra*. Approximate formulas are

$$\begin{aligned}x(t) &\approx \int_0^t \left(1.10e^{1.6(t-u)} - 0.12e^{-2.6(t-u)}\right) r(u) du, \\y(t) &\approx \int_0^t \left(0.49e^{1.6(t-u)} - 0.49e^{-2.6(t-u)}\right) r(u) du.\end{aligned}$$

The exponential rates 1.6 and  $-2.6$  represent respectively the accumulation of the pesticide into the soil and the decay of the pesticide from the trees. The application rate  $r(t)$  is typically a step function equal to a positive constant over a small interval of time and zero elsewhere, or a sum of such functions, representing periodic applications of pesticide.

## Forecasting Prices

A cosmetics manufacturer has a marketing policy based upon the price  $x(t)$  of its salon shampoo.



**Figure 13. Salon shampoo sample.**

The marketing strategy for the shampoo is to set the price  $x(t)$  dynamically to reflect demand for the product.

The **production**  $P(t)$  and the **sales**  $S(t)$  are given in terms of the **price**  $x(t)$  and the **change in price**  $x'(t)$  by the equations

$$P(t) = 4 - \frac{3}{4}x(t) - 8x'(t) \quad (\text{Production}),$$

$$S(t) = 15 - 4x(t) - 2x'(t) \quad (\text{Sales}).$$

The differential equations for the price  $x(t)$  and inventory level  $I(t)$  are

$$\begin{aligned} x'(t) &= k(I(t) - I_0), \\ I'(t) &= P(t) - S(t). \end{aligned}$$

The inventory level  $I_0 = 50$  represents the desired level. The equations can be written in terms of  $x(t)$ ,  $I(t)$  as follows.

$$\begin{aligned} x'(t) &= kI(t) - kI_0, \\ I'(t) &= \frac{13}{4}x(t) - 6kI(t) + 6kI_0 - 11. \end{aligned}$$

If  $k = 1$ ,  $x(0) = 10$  and  $I(0) = 7$ , then the solution is given by

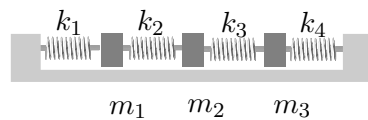
$$\begin{aligned} x(t) &= \frac{44}{13} + \frac{86}{13}e^{-13t/2}, \\ I(t) &= 50 - 43e^{-13t/2}. \end{aligned}$$

The **forecast** of price  $x(t) \approx 3.39$  dollars at inventory level  $I(t) \approx 50$  is based upon the two limits

$$\lim_{t \rightarrow \infty} x(t) = \frac{44}{13}, \quad \lim_{t \rightarrow \infty} I(t) = 50.$$

## Coupled Spring-Mass Systems

Three masses are attached to each other by four springs as in Figure 14.



**Figure 14. Three masses connected by springs.** The masses slide along a frictionless horizontal surface.

The analysis uses the following constants, variables and assumptions.

<b>Mass</b>	The masses $m_1$ , $m_2$ , $m_3$ are assumed to be point masses concentrated at their center of gravity.
<b>Constants</b>	
<b>Spring</b>	The mass of each spring is negligible. The springs operate according to Hooke's law: Force = $k$ (elongation).
<b>Constants</b>	Constants $k_1$ , $k_2$ , $k_3$ , $k_4$ denote the Hooke's constants. The springs restore after compression and extension.
<b>Position</b>	The symbols $x_1(t)$ , $x_2(t)$ , $x_3(t)$ denote the mass positions along the horizontal surface, measured from their equilibrium positions, plus right and minus left.
<b>Variables</b>	



**Fixed Ends** The first and last spring are attached to fixed walls.

The **competition method** is used to derive the equations of motion. In this case, the law is

Newton's Second Law Force = Sum of the Hooke's Forces.

The model equations are

$$(6) \quad \begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) + k_2 [x_2(t) - x_1(t)], \\ m_2 x_2''(t) &= -k_2 [x_2(t) - x_1(t)] + k_3 [x_3(t) - x_2(t)], \\ m_3 x_3''(t) &= -k_3 [x_3(t) - x_2(t)] - k_4 x_3(t). \end{aligned}$$

The equations are justified in the case of all positive variables by observing that the first three springs are elongated by  $x_1$ ,  $x_2 - x_1$ ,  $x_3 - x_2$ , respectively. The last spring is compressed by  $x_3$ , which accounts for the minus sign.

Another way to justify the equations is through mirror-image symmetry: interchange  $k_1 \longleftrightarrow k_4$ ,  $k_2 \longleftrightarrow k_3$ ,  $x_1 \longleftrightarrow x_3$ , then equation 2 should be unchanged and equation 3 should become equation 1.

**Matrix Formulation.** System (6) can be written as a second order vector-matrix system

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

More succinctly, the system is written as

$$M\mathbf{x}''(t) = K\mathbf{x}(t)$$

where the **displacement**  $\mathbf{x}$ , **mass matrix**  $M$  and **stiffness matrix**  $K$  are defined by the formulas

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

**Numerical example.** Let  $m_1 = 1$ ,  $m_2 = 1$ ,  $m_3 = 1$ ,  $k_1 = 2$ ,  $k_2 = 1$ ,  $k_3 = 1$ ,  $k_4 = 2$ . Then the system is given by

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

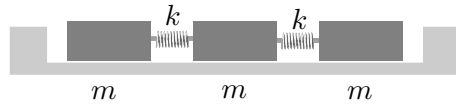
The vector solution is given by the formula

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= (a_1 \cos t + b_1 \sin t) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ &+ (a_2 \cos \sqrt{3}t + b_2 \sin \sqrt{3}t) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &+ (a_3 \cos 2t + b_3 \sin 2t) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \end{aligned}$$

where  $a_1, a_2, a_3, b_1, b_2, b_3$  are arbitrary constants.

## Boxcars

A special case of the coupled spring-mass system is three boxcars on a level track connected by springs, as in Figure 15.



**Figure 15. Three identical boxcars connected by identical springs.**

Except for the springs on fixed ends, this problem is the same as the one of the preceding illustration, therefore taking  $k_1 = k_4 = 0, k_2 = k_3 = k, m_1 = m_2 = m_3 = m$  gives the system

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Take  $k/m = 1$  to obtain the illustration

$$\mathbf{x}'' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x},$$

which has vector solution

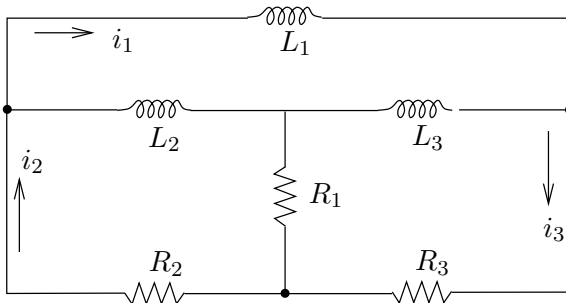
$$\begin{aligned} \mathbf{x} &= (a_1 + b_1 t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (a_2 \cos t + b_2 \sin t) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &+ (a_3 \cos \sqrt{3}t + b_3 \sin \sqrt{3}t) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \end{aligned}$$

where  $a_1, a_2, a_3, b_1, b_2, b_3$  are arbitrary constants.

The solution expression can be used to discover what happens to the boxcars when the springs act normally upon compression but disengage upon expansion. An interesting physical situation is when one car moves along the track, contacts two stationary cars, then transfers its momentum to the other cars, followed by disengagement.

## Electrical Network I

Consider the  $LR$ -network of Figure 16.



**Figure 16. An electrical network.** There are three resistors  $R_1, R_2, R_3$  and three inductors  $L_1, L_2, L_3$ . The currents  $i_1, i_2, i_3$  are defined between nodes (black dots).

The derivation of the differential equations for the loop currents  $i_1, i_2, i_3$  uses Kirchhoff's laws and the voltage drop formulas for resistors and inductors. The black dots in the diagram are the **nodes** that determine the beginning and end of each of the currents  $i_1, i_2, i_3$ . Currents are defined only on the outer boundary of the network. Kirchhoff's node law determines the currents across  $L_2, L_3$  (arrowhead right) as  $i_2 - i_1$  and  $i_3 - i_1$ , respectively. Similarly,  $i_2 - i_3$  is the current across  $R_1$  (arrowhead down). Using Ohm's law  $V_R = RI$  and Faraday's law  $V_L = LI'$  plus Kirchhoff's loop law *algebraic sum of the voltage drops is zero* around a closed loop (see the `maple` code below), we arrive at the model

$$\begin{aligned} i_1' &= - \left( \frac{R_2}{L_1} \right) i_2 - \left( \frac{R_3}{L_1} \right) i_3, \\ i_2' &= - \left( \frac{R_2}{L_2} + \frac{R_2}{L_1} \right) i_2 + \left( \frac{R_1}{L_2} - \frac{R_3}{L_1} \right) i_3, \\ i_3' &= \left( \frac{R_1}{L_3} - \frac{R_2}{L_1} \right) i_2 - \left( \frac{R_1}{L_3} + \frac{R_3}{L_1} + \frac{R_3}{L_3} \right) i_3 \end{aligned}$$

A computer algebra system is helpful to obtain the differential equations from the closed loop formulas. Part of the theory is that the number of equations equals the number of *holes* in the network, called the **connectivity**. Here's some `maple` code for determining the equations in scalar and also in vector-matrix form.

```
loop1:=L1*D(i1)+R3*i3+R2*i2=0;
loop2:=L2*D(i2)-L2*D(i1)+R1*(i2-i3)+R2*i2=0;
loop3:=L3*D(i3)-L3*D(i1)+R3*i3+R1*(i3-i2)=0;
```

```

f1:=solve(loop1,D(i1));
f2:=solve(subs(D(i1)=f1,loop2),D(i2));
f3:=solve(subs(D(i1)=f1,loop3),D(i3));
with(linalg):
jacobian([f1,f2,f3],[i1,i2,i3]);

```

## Electrical Network II

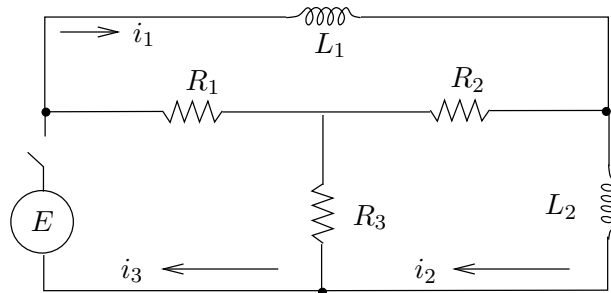
Consider the  $LR$ -network of Figure 17. This network produces only two differential equations, even though there are three *holes* (connectivity 3). The derivation of the differential equations parallels the previous network, so nothing will be repeated here.

A computer algebra system is used to obtain the differential equations from the closed loop formulas. Below is `maple` code to generate the equations  $i'_1 = f_1$ ,  $i'_2 = f_2$ ,  $i_3 = f_3$ .

```

loop1:=L1*D(i1)+R2*(i1-i2)+R1*(i1-i3)=0;
loop2:=L2*D(i2)+R3*(i2-i3)+R2*(i2-i1)=0;
loop3:=R3*(i3-i2)+R1*(i3-i1)=E;
f3:=solve(loop3,i3);
f1:=solve(subs(i3=f3,loop1),D(i1));
f2:=solve(subs(i3=f3,loop2),D(i2));

```



**Figure 17.** An electrical network.

There are three resistors  $R_1$ ,  $R_2$ ,  $R_3$ , two inductors  $L_1$ ,  $L_2$  and a battery  $E$ . The currents  $i_1$ ,  $i_2$ ,  $i_3$  are defined between nodes (black dots).

The model, in the special case  $L_1 = L_2 = 1$  and  $R_1 = R_2 = R_3 = R$ :

$$\begin{aligned}
 i'_1 &= -\frac{3R}{2}i_1 + \frac{3R}{2}i_2 + \frac{E}{2}, \\
 i'_2 &= \frac{3R}{2}i_1 - \frac{3R}{2}i_2 + \frac{E}{2}, \\
 i_3 &= \frac{1}{2}i_1 + \frac{1}{2}i_2 + \frac{E}{2R}.
 \end{aligned}$$

It is easily justified that the solution of the differential equations for

initial conditions  $i_1(0) = i_2(0) = 0$  is given by

$$i_1(t) = \frac{E}{2}t, \quad i_2(t) = \frac{E}{2}t.$$

### Logging Timber by Helicopter

Certain sections of National Forest in the USA do not have logging access roads. In order to log the timber in these areas, helicopters are employed to move the felled trees to a nearby loading area, where they are transported by truck to the mill. The felled trees are slung beneath the helicopter on cables.

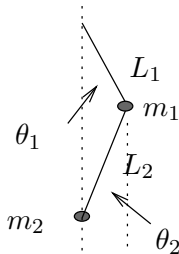


**Figure 18. Helicopter logging.**  
**Left:** An Erickson helicopter lifts felled trees.  
**Right:** Two trees are attached to the cable to lower transportation costs.

The payload for two trees approximates a double pendulum, which oscillates during flight. The angles of oscillation  $\theta_1, \theta_2$  of the two connecting cables, measured from the gravity vector direction, satisfy the following differential equations, in which  $g$  is the gravitation constant,  $m_1, m_2$  denote the masses of the two trees and  $L_1, L_2$  are the cable lengths.

$$\begin{aligned} (m_1 + m_2)L_1^2\theta_1'' + m_2L_1L_2\theta_2'' + (m_1 + m_2)L_1g\theta_1 &= 0, \\ m_2L_1L_2\theta_1'' + m_2L_2^2\theta_2'' + m_2L_2g\theta_2 &= 0. \end{aligned}$$

This model is derived assuming small displacements  $\theta_1, \theta_2$ , that is,  $\sin \theta \approx \theta$  for both angles, using the following diagram.



**Figure 19. Logging Timber by Helicopter.**  
 The cables have lengths  $L_1, L_2$ . The angles  $\theta_1, \theta_2$  are measured from vertical.

The lengths  $L_1, L_2$  are adjusted on each trip for the length of the trees, so that the trees do not collide in flight with each other nor with the helicopter. Sometimes, three or more smaller trees are bundled together in a package, which is treated here as identical to a single, very thick tree hanging on the cable.

**Vector-matrix model.** The angles  $\theta_1, \theta_2$  satisfy the second-order

vector-matrix equation

$$\begin{pmatrix} (m_1 + m_2)L_1 & m_2L_2 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}'' = - \begin{pmatrix} m_1g + m_2g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

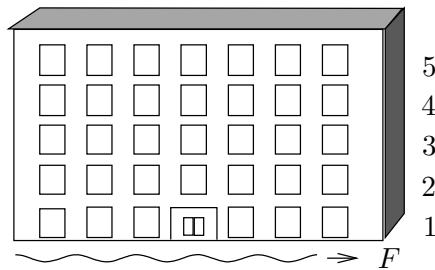
This system is equivalent to the second-order system

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}'' = \begin{pmatrix} -\frac{m_1g + m_2g}{L_1m_1} & \frac{m_2g}{L_1m_1} \\ \frac{m_1g + m_2g}{L_2m_1} & -\frac{(m_1 + m_2)g}{L_2m_1} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

## Earthquake Effects on Buildings

A horizontal earthquake oscillation  $F(t) = F_0 \cos \omega t$  affects each floor of a 5-floor building; see Figure 20. The effect of the earthquake depends upon the natural frequencies of oscillation of the floors.

In the case of a single-floor building, the center-of-mass position  $x(t)$  of the building satisfies  $mx'' + kx = E$  and the natural frequency of oscillation is  $\sqrt{k/m}$ . The earthquake force  $E$  is given by Newton's second law:  $E(t) = -mF''(t)$ . If  $\omega \approx \sqrt{k/m}$ , then the amplitude of  $x(t)$  is large compared to the amplitude of the force  $E$ . The amplitude increase in  $x(t)$  means that a small-amplitude earthquake wave can resonant with the building and possibly demolish the structure.



**Figure 20. A 5-Floor Building.**

A horizontal earthquake wave  $F$  affects every floor. The actual wave has wavelength many times larger than the illustration.

The following assumptions and symbols are used to quantize the oscillation of the 5-floor building.

- Each floor is considered a point mass located at its center-of-mass. The floors have masses  $m_1, \dots, m_5$ .
- Each floor is restored to its equilibrium position by a linear restoring force or Hooke's force  $-k(\text{elongation})$ . The Hooke's constants are  $k_1, \dots, k_5$ .
- The locations of masses representing the 5 floors are  $x_1, \dots, x_5$ . The equilibrium position is  $x_1 = \dots = x_5 = 0$ .

- Damping effects of the floors are ignored. This is a *frictionless* system.

The differential equations for the model are obtained by **competition**: the Newton's second law force is set equal to the sum of the Hooke's forces and the external force due to the earthquake wave. This results in the following system, where  $k_6 = 0$ ,  $E_j = m_j F''$  for  $j = 1, 2, 3, 4, 5$  and  $F = F_0 \cos \omega t$ .

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + E_1, \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3 + E_2, \\ m_3 x_3'' &= k_3 x_2 - (k_3 + k_4)x_3 + k_4 x_4 + E_3, \\ m_4 x_4'' &= k_4 x_3 - (k_4 + k_5)x_4 + k_5 x_5 + E_4, \\ m_5 x_5'' &= k_5 x_4 - (k_5 + k_6)x_5 + E_5. \end{aligned}$$

In particular, the equations for a floor depend only upon the neighboring floors. The bottom floor and the top floor are exceptions: they have just one neighboring floor.

**Vector-matrix second order system.** Define

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{pmatrix},$$

$$K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 & 0 & 0 \\ k_2 & -k_2 - k_3 & k_3 & 0 & 0 \\ 0 & k_3 & -k_3 - k_4 & k_4 & 0 \\ 0 & 0 & k_4 & -k_4 - k_5 & k_5 \\ 0 & 0 & 0 & k_5 & -k_5 - k_6 \end{pmatrix}.$$

In the last row,  $k_6 = 0$ , to reflect the absence of a floor above the fifth. The second order system is

$$M\mathbf{x}''(t) = K\mathbf{x}(t) + \mathbf{H}(t).$$

The matrix  $M$  is called the **mass matrix** and the matrix  $K$  is called the **Hooke's matrix**. The **external force**  $\mathbf{H}(t)$  can be written as a scalar function  $E(t) = -F''(t)$  times a constant vector:

$$\mathbf{H}(t) = -\omega^2 F_0 \cos \omega t \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}.$$

**Identical floors.** Let us assume that all floors have the same mass  $m$  and the same Hooke's constant  $k$ . Then  $M = mI$  and the equation becomes

$$\mathbf{x}'' = m^{-1} \begin{pmatrix} -2k & k & 0 & 0 & 0 \\ k & -2k & k & 0 & 0 \\ 0 & k & -2k & k & 0 \\ 0 & 0 & k & -2k & k \\ 0 & 0 & 0 & k & -k \end{pmatrix} \mathbf{x} - F_0 \omega^2 \cos(\omega t) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The Hooke's matrix  $K$  is symmetric ( $K^T = K$ ) with negative entries only on the diagonal. The last diagonal entry is  $-k$  (a common error is to write  $-2k$ ).

**Particular solution.** The method of undetermined coefficients predicts a trial solution  $\mathbf{x}_p(t) = \mathbf{c} \cos \omega t$ , because each differential equation has nonhomogeneous term  $-F_0 \omega^2 \cos \omega t$ . The constant vector  $\mathbf{c}$  is found by trial solution substitution. Cancel the common factor  $\cos \omega t$  in the substituted equation to obtain the equation  $(m^{-1}K + \omega^2 I) \mathbf{c} = F_0 \omega^2 \mathbf{b}$ , where  $\mathbf{b}$  is column vector of ones in the preceding display. Let  $B(\omega) = m^{-1}K + \omega^2 I$ . Then the formula  $B^{-1} = \frac{\mathbf{adj}(B)}{\det(B)}$  gives

$$\mathbf{c} = F_0 \omega^2 \frac{\mathbf{adj}(B(\omega))}{\det(B(\omega))} \mathbf{b}.$$

The constant vector  $\mathbf{c}$  can have a large magnitude when  $\det(B(\omega)) \approx 0$ . This occurs when  $-\omega^2$  is nearly an eigenvalue of  $m^{-1}K$ .

**Homogeneous solution.** The theory of this chapter gives the homogeneous solution  $\mathbf{x}_h(t)$  as the sum

$$\mathbf{x}_h(t) = \sum_{j=1}^5 (a_j \cos \omega_j t + b_j \sin \omega_j t) \mathbf{v}_j$$

where  $r = \omega_j$  and  $\mathbf{v} = \mathbf{v}_j \neq \mathbf{0}$  satisfy

$$\left( \frac{1}{m} K + r^2 I \right) \mathbf{v} = \mathbf{0}.$$

**Special case**  $k/m = 10$ . Then

$$\frac{1}{m} K = \begin{pmatrix} -20 & 10 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 10 & -10 \end{pmatrix}$$



and the values  $\omega_1, \dots, \omega_5$  are found by solving the determinant equation  $\det((1/m)K + \omega^2 I) = 0$ , to obtain the values in Table 1.

**Table 1.** The natural frequencies for the special case  $k/m = 10$ .

Frequency	Value
$\omega_1$	0.900078068
$\omega_2$	2.627315231
$\omega_3$	4.141702938
$\omega_4$	5.320554507
$\omega_5$	6.068366391

**General solution.** Superposition implies  $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$ . Both terms of the general solution represent bounded oscillations.

**Resonance effects.** The special solution  $\mathbf{x}_p(t)$  can be used to obtain some insight into practical resonance effects between the incoming earthquake wave and the building floors. When  $\omega$  is close to one of the frequencies  $\omega_1, \dots, \omega_5$ , then the amplitude of a component of  $\mathbf{x}_p$  can be very large, causing the floor to take an excursion that is too large to maintain the structural integrity of the floor.

The **physical interpretation** is that an earthquake wave of the proper frequency, having time duration sufficiently long, can demolish a floor and hence demolish the entire building. The amplitude of the earthquake wave does not have to be large: a fraction of a centimeter might be enough to start the oscillation of the floors.

## Earthquakes and Tsunamis

The original **Richter scale**, with depreciated use in seismology, was invented by seismologist C. Richter to rate earthquake power.

The moment magnitude scale ( $M_W$ ) has largely replaced the original Richter scale and its modified versions. The highest reported magnitude is 9.5  $M_W$  by the United States Geological Survey for the Concepción, Chile earthquake of May 22, 1960. News reports and the general public still refer to earthquake magnitude using the term *Richter Scale*.

The Sumatra earthquake of December 26, 2004 occurred close to a deep-sea trench, a subduction zone where one tectonic plate slips beneath another. Most of the earthquake energy is released in these areas as the two plates grind towards each other. Estimates of magnitude 8.8  $M_W$  to 9.3  $M_W$  followed the event. The US Geological Survey estimated 9.2  $M_W$ .

The Chile earthquake and tsunami of 1960 has been documented well. Here is an account by Dr. Gerard Fryer of the Hawaii Institute of Geophysics and Planetology, in Honolulu.

The tsunami was generated by the Chile earthquake of May 22, 1960, the largest earthquake ever recorded: it was magnitude 9.6. What happened in the earthquake was that a piece of the Pacific seafloor (or strictly speaking, the Nazca Plate) about the size of California slid fifty feet beneath the continent of South America. Like a spring, the lower slopes of the South American continent offshore snapped upwards as much as twenty feet while land along the Chile coast dropped about ten feet. This change in the shape of the ocean bottom changed the shape of the sea surface. Since the sea surface likes to be flat, the pile of excess water at the surface collapsed to create a series of waves — the tsunami.

The tsunami, together with the coastal subsidence and flooding, caused tremendous damage along the Chile coast, where about 2,000 people died. The waves spread outwards across the Pacific. About 15 hours later the waves flooded Hilo, on the island of Hawaii, where they built up to 30 feet and caused 61 deaths along the waterfront. Seven hours after that, 22 hours after the earthquake, the waves flooded the coastline of Japan where 10-foot waves caused 200 deaths. The waves also caused damage in the Marquesas, in Samoa, and in New Zealand. Tidal gauges throughout the Pacific measured anomalous oscillations for about three days as the waves bounced from one side of the ocean to the other.