Sample Quiz 4

Sample Quiz 4, Problem 1. Picard's Theorem and RLC-Circuit Models

Picard-Lindelöf Theorem. Let $\vec{f}(x, \vec{y})$ be defined for $|x - x_0| \le h$, $\|\vec{y} - \vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &=& \vec{f}(x, \vec{y}(x)), \quad |x - x_0| < H, \\ \vec{y}(x_0) &=& \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x - x_0| < H$.

Emile Picard

The Problem. The second order problem

$$\begin{cases} u'' + 2u' + 5u = 60\sin(20x), \\ u(0) = 1, \\ u'(0) = 0 \end{cases} \tag{1}$$

is an RLC-circuit charge model, in which the variables have been changed. The variables are time x in seconds and charge $u(x)$ in coulombs. Coefficients in the equation represent an inductor $L = 1$ H, a resistor $R = 2\,\Omega$, a capacitor $C = 0.2$ F and a voltage input $E(x) = 60\sin(20x)$.

The several parts below detail how to convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies. Please fill in the missing details.

(a) The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where y_1, y_2 are the invented components of vector $\vec{y}$. Then the initial data $u(0) = 1, u'(0) = 0$ converts to the vector initial data

$$\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -5y_1 - 2y_2 + 60\sin(20x).$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$\begin{cases} \vec{y}\,'(x) &=& \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}, \\ \vec{y}(0) &=& \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases} \tag{2}$$

(d) System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables x, y_1, y_2, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

Solutions to Problem 1

(a) $\vec{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

(b) Differentiate, $y_1' = u'(x) = y_2$ and $y_2' = u''(x)$. Isolate u'' left in the equation $u''+2u'+5u = 60\sin(20x)$, then reduce $y_2' = u''(x)$ into $y_2' = -2u' - 5u + 60\sin(20x) = -2y_2 - 5y_1 + 60\sin(20x)$.

(c) Initial data $\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ was derived in part (a). The differential equation is derived from the scalar dynamical system in part (b), as follows.

$$\begin{aligned}
\vec{y}\,' &= \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 + 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}
\end{aligned}$$

(d) From calculus, polynomials and trigonometric function sine are infinitely differentiable. Therefore, in each of the variables x, y_1, y_2 the components of $\vec{f}$, which are just the right sides of the dynamical system equations of part (b), are also infinitely differentiable.

Sample Quiz4 Problem 2. The velocity of a crossbow arrow fired upward from the ground is given at different times in the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	50	Ground
1.413	0	Maximum
2.980	-45	Near Ground Impact

(a) The velocity can be approximated by a quadratic polynomial

$$v(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c. Then solve for them to obtain $a \approx 2.238$, $b \approx -38.55$, $c = 50$.

(b) Assume a drag model $v' = -32 - \rho v$. Substitute the polynomial answer of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.131$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 50$ with $\rho = 0.131$.

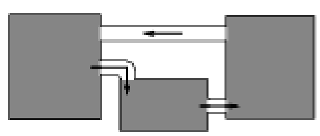
(d) Compare $w(t)$ and $v(t)$ in a plot. Comment on the plot and what it means.

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

Sample Quiz4 Extra Credit Problem 3. Consider the system of differential equations

$$\begin{aligned} x_1' &= -\tfrac{1}{6}x_1 & & + \tfrac{1}{6}x_3, \\ x_2' &= \tfrac{1}{6}x_1 - \tfrac{1}{3}x_2, & & \\ x_3' &= \tfrac{1}{3}x_2 - \tfrac{1}{6}x_3, & & \end{aligned}$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C
The volumes are $60, 30, 60$ for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols x_1, x_2, x_3. Solve the corresponding linear system of algebraic equations to obtain the answer $x_1 = x_3 = 2c$, $x_2 = c$, which means the total amount of salt is uniformly distributed in the tanks in ratio $2 : 1 : 2$.

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.