Sample Quiz 4

Sample Quiz 4, Problem 1. Picard’s Theorem and RLC-Circuit Models

Picard-Lindelöf Theorem. Let \( \vec{f}(x, \vec{y}) \) be defined for \(|x - x_0| \leq h, \|\vec{y} - \vec{y}_0\| \leq k \), with \( \vec{f} \) and \( \frac{\partial \vec{f}}{\partial \vec{y}} \) continuous. Then for some constant \( H, 0 < H < h \), the problem

\[
\begin{cases}
\vec{y}'(x) = \vec{f}(x, \vec{y}(x)), & |x - x_0| < H, \\
\vec{y}(x_0) = \vec{y}_0
\end{cases}
\]

has a unique solution \( \vec{y}(x) \) defined on the smaller interval \(|x - x_0| < H\).

The Problem. The second order problem

\[
\begin{cases}
\frac{d^2 u}{dx^2} + 2 \frac{du}{dx} + 5u = 60 \sin(20x), \\
u(0) = 1, \\
u'(0) = 0
\end{cases}
\]

(1)

is an RLC-circuit charge model, in which the variables have been changed. The variables are time \( x \) in seconds and charge \( u(x) \) in coulombs. Coefficients in the equation represent an inductor \( L = 1 \text{ H} \), a resistor \( R = 2 \text{ Ω} \), a capacitor \( C = 0.2 \text{ F} \) and a voltage input \( E(x) = 60 \sin(20x) \).

The several parts below detail how to convert the scalar initial value problem into a vector problem, to which Picard’s vector theorem applies. Please fill in the missing details.

(a) The conversion uses the position-velocity substitution \( y_1 = u(x), y_2 = u'(x) \), where \( y_1, y_2 \) are the invented components of vector \( \vec{y} \). Then the initial data \( u(0) = 1, u'(0) = 0 \) converts to the vector initial data

\[
\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

(b) Differentiate the equations \( y_1 = u(x), y_2 = u'(x) \) in order to find the scalar system of two differential equations, known as a dynamical system:

\[
y_1' = y_2, \quad y_2' = -5y_1 - 2y_2 + 60 \sin(20x).
\]

(c) The derivative of vector function \( \vec{y}(x) \) is written \( \vec{y}'(x) \) or \( \frac{d\vec{y}}{dx}(x) \). It is obtained by componentwise differentiation: \( \vec{y}'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \). The vector differential equation model of scalar system (1) is

\[
\begin{cases}
\vec{y}'(x) = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 60 \sin(20x) \end{pmatrix}, \\
\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{cases}
\]

(2)

(d) System (2) fits the hypothesis of Picard’s theorem, using symbols

\[
\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 60 \sin(20x) \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

The components of vector function \( \vec{f} \) are continuously differentiable in variables \( x, y_1, y_2 \), therefore \( \vec{f} \) and \( \frac{\partial \vec{f}}{\partial \vec{y}} \) are continuous.
Solutions to Problem 1

(a) \( \vec{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

(b) Differentiate, \( y'_1 = u'(x) = y_2 \) and \( y'_2 = u''(x) \). Isolate \( u'' \) left in the equation \( u'' + 2u' + 5u = 60 \sin(20x) \), then reduce \( y'_2 = u''(x) \) into \( y'_2 = -2u' - 5u + 60 \sin(20x) = -2y_2 - 5y_1 + 60 \sin(20x) \).

(c) Initial data \( \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) was derived in part (a). The differential equation is derived from the scalar dynamical system in part (b), as follows.

\[
\begin{aligned}
\vec{y}' &= \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 + 60 \sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60 \sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 60 \sin(20x) \end{pmatrix} \\
\end{aligned}
\]

(d) From calculus, polynomials and trigonometric function sine are infinitely differentiable. Therefore, in each of the variables \( x, y_1, y_2 \) the components of \( \vec{f} \), which are just the right sides of the dynamical system equations of part (b), are also infinitely differentiable.
Sample Quiz4 Problem 2. The velocity of a crossbow arrow fired upward from the ground is given at different times in the following table.

<table>
<thead>
<tr>
<th>Time $t$ in seconds</th>
<th>Velocity $v(t)$ in ft/sec</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>50</td>
<td>Ground</td>
</tr>
<tr>
<td>1.413</td>
<td>0</td>
<td>Maximum</td>
</tr>
<tr>
<td>2.980</td>
<td>-45</td>
<td>Near Ground Impact</td>
</tr>
</tbody>
</table>

(a) The velocity can be approximated by a quadratic polynomial

$$v(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients $a, b, c$. Then solve for them to obtain $a \approx 2.238$, $b \approx -38.55$, $c = 50$.

(b) Assume a drag model $v' = -32 - \rho v$. Substitute the polynomial answer of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.131$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 50$ with $\rho = 0.131$.

(d) Compare $w(t)$ and $v(t)$ in a plot. Comment on the plot and what it means.

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

Sample Quiz4 Extra Credit Problem 3. Consider the system of differential equations

\[
\begin{align*}
x'_1 &= -\frac{1}{6}x_1 + \frac{1}{6}x_3, \\
x'_2 &= \frac{1}{6}x_1 - \frac{1}{3}x_2, \\
x'_3 &= \frac{1}{3}x_2 - \frac{1}{6}x_3,
\end{align*}
\]

for the amounts $x_1, x_2, x_3$ of salt in recirculating brine tanks, as in the figure:

Recirculating Brine Tanks A, B, C

The volumes are 60, 30, 60 for $A, B, C$, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x'_1 = x'_2 = x'_3 = 0$ and then solving for the symbols $x_1, x_2, x_3$. Solve the corresponding linear system of algebraic equations to obtain the answer $x_1 = x_3 = 2c$, $x_2 = c$, which means the total amount of salt is uniformly distributed in the tanks in ratio 2 : 1 : 2.

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.