Solutions to Problem 1

(a) $\vec{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) Differentiate, $y_1' = u'(x) = y_2$ and $y_2' = u''(x)$. Isolate u'' left in the equation $u''+2u'+5u = 60\sin(20x)$, then reduce $y_2' = u''(x)$ into $y_2' = -2u' - 5u + 60\sin(20x) = -2y_2 - 5y_1 + 60\sin(20x)$.

(c) Initial data $\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ was derived in part (a). The differential equation is derived from the scalar dynamical system in part (b), as follows.

$$\begin{aligned}
\vec{y}\,' &= \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 + 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} y_2 \\ -5y_1 - 2y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}\vec{y} + \begin{pmatrix} 0 \\ 60\sin(20x) \end{pmatrix}
\end{aligned}$$

(d) From calculus, polynomials and trigonometric function sine are infinitely differentiable. Therefore, in each of the variables x, y_1, y_2 the components of $\vec{f}$, which are just the right sides of the dynamical system equations of part (b), are also infinitely differentiable.

Sample Quiz 4 Solutions

problem 2

(a) Let $t_1 = 1.413$, $t_2 = 2.98$. Use $at^2+bt+c = v(t)$ for $t=0, t_1, t_2$ to obtain the system

$$\begin{cases} a\cdot 0^2 + b\cdot 0 + c = 50 \\ a\cdot t_1^2 + b\cdot t_1 + c = 0 \\ a\cdot t_2^2 + b t_2 + c = -45 \end{cases}$$

Then $\boxed{c=50}$. The 3X3 system reduces to a 2x2 system

$$\begin{cases} a t_1^2 + b t_1 = -50 \\ a t_2^2 + b t_2 = -95 \end{cases}$$

$$\begin{cases} a + b/t_1 = -50/t_1^2 \\ a t_2^2 + b t_2 = -95 \end{cases} \qquad \text{mult}(1, 1/t_1^2)$$

$$\begin{cases} a + b/t_1 = -50\, t_1^2 \\ 0 + b\cdot t_3 = -95 + \frac{50 t_2^2}{t_1^2} \end{cases} \qquad \text{combo}(1,2,-t_2^2) \text{ where } t_3 = t_2 - \frac{t_2^2}{t_1}$$

Then $\boxed{b} = \frac{1}{t_2-t_1}\left(\frac{-50 t_2}{t_1} + 95\frac{t_1}{t_2}\right) = \boxed{-38.54760463}$

$\boxed{a} = \frac{1}{t_2-t_1}\left(\frac{50}{t_1} - \frac{95}{t_2}\right) = \boxed{2.23772148}$

The example gives evidence for why technology is used on systems of equations. For 2-digit accuracy, it is less hand work. Also quite fast with a calculator.

(b) Substitution gives $2at + b = -32 - \rho(at^2+bt+c)$, Then $t=0$ implies $b = -32 - \rho c$. Calculator gives $\rho = (-32-b)/c = 0.1309520926 \cong 0.131$

(c) $w = \text{equil sol} + \frac{c_1}{\text{integ factor}} = \frac{-32}{\rho} + \frac{c_1}{e^{\rho t}}$. Then $w(0)=50$ implies $c_1 = 50 + 32/\rho$.

(d) A good plot is $|v(t)-w(t)|$ on $0 \le t \le 3$. It shows max error of 0.3.

Sample Quiz 4 Solutions

__problem 3__ System $\begin{cases} -\frac{1}{6}x_1 + \frac{1}{6}x_3 = 0 \\ \frac{1}{6}x_1 - \frac{1}{3}x_2 = 0 \\ \frac{1}{3}x_2 - \frac{1}{6}x_3 = 0 \end{cases}$ has augmented matrix equal to

$$\left(\begin{array}{ccc|c} -\frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \quad \text{mult}(1,6), \text{mult}(2,6), \text{mult}(3,6)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \quad \text{combo}(1,2,1)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{Combo}(2,3,1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{mult}(1,-1), \text{mult}(2,-1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{mult}(2, \tfrac{1}{2}) \qquad \text{Last frame}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \\ 0 = 0 \end{cases}$$

reduced echelon system
last frame algorithm applies

__Answer__

$$\begin{cases} x_1 = t_1 \\ x_2 = \frac{1}{2}t_1 \\ x_3 = t_1 \end{cases} \qquad -\infty < t_1 < \infty$$

__Let $t_1 = 2c$__
Then
$$\begin{cases} x_1 = 2c \\ x_2 = c \\ x_3 = 2c \end{cases}$$

Salt amounts in steady-state are in ratio 2:1:2