Sample Quiz 11

Background. Switches and Impulses

Laplace’s method solves differential equations. It is the preferred method for solving equations containing switches or impulses.

Unit Step  Define \( u(t - a) = \begin{cases} 
1 & t \geq a, \\
0 & t < a. 
\end{cases} \). It is a switch, turned on at \( t = a \).

Ramp  Define \( \text{ramp}(t - a) = (t - a)u(t - a) = \begin{cases} 
\frac{t - a}{b - a} & t \geq a, \\
0 & t < a. 
\end{cases} \), whose graph shape is a continuous ramp at 45-degree incline starting at \( t = a \).

Unit Pulse  Define \( \text{pulse}(t, a, b) = \begin{cases} 
1 & a \leq t < b, \\
0 & \text{otherwise} 
\end{cases} = u(t - a) - u(t - b) \). The switch is ON at time \( t = a \) and then OFF at time \( t = b \).

Impulse of a Force

Define the impulse of an applied force \( F(t) \) on time interval \( a \leq t \leq b \) by the equation

\[
\text{Impulse of } F = \int_{a}^{b} F(t) dt = \left( \frac{\int_{a}^{b} F(t) dt}{b - a} \right) (b - a) = \text{Average Force} \times \text{Duration Time}.
\]

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the impulse of the force is deemed important, and not its magnitude nor duration.

Define the Dirac Unit Impulse by the equation \( \delta(t - a) = \frac{du}{dt}(t - a) \), where \( u(t - a) \) is the unit step. Symbol \( \delta \) makes sense only under an integral sign, and the integral in question must be a generalized Riemann integral (definition pending), with new evaluation rules. Symbol \( \delta \) is an abbreviation like etc or e.g., because it abbreviates a paragraph of descriptive text.

- Symbol \( M\delta(t - a) \) represents an ideal impulse of magnitude \( M \) at time \( t = a \). Value \( M \) is the change in momentum, but \( M\delta(t - a) \) contains no detail about the applied force or the duration.
  A common force approximation for a hammer hit of very small duration \( 2h \) and impulse \( M \) is Dirac’s approximation

\[
F_h(t) = \frac{M}{2h} \text{pulse}(t, a - h, a + h).
\]

- Symbol \( \delta \) is not manipulated as an ordinary function. It is a special modeling tool with rules for application and rules for algebraic manipulation.

**THEOREM** (Second Shifting Theorem). Let \( f(t) \) and \( g(t) \) be piecewise continuous and of exponential order. Then for \( a \geq 0 \),

\[
e^{-as} \mathcal{L}(f(t)) = \mathcal{L} \left( f(t)u(t) |_{t=a} \right),
\]

\[
\mathcal{L}(g(t)u(t - a)) = e^{-as} \mathcal{L} \left( g(t) |_{t=a} \right).
\]
Problem 1. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for the unit step, ramp and pulse, in these special cases:

1. \( \mathcal{L}(10u(t - \pi)) \)
2. \( \mathcal{L}(\text{ramp}(t - 2\pi)) \)
3. \( \mathcal{L}(10 \text{pulse}(t, 3, 5)) \).

(b) Backward table. Find \( f(t) \) in the following special cases.

1. \( \mathcal{L}(f) = \frac{5e^{-3s}}{s} \)
2. \( \mathcal{L}(f) = \frac{e^{-4s}}{s^2} \)
3. \( \mathcal{L}(f) = \frac{5}{s} \left( e^{-2s} - e^{-3s} \right) \).
Problem 2. Solve the following Dirac impulse problems.

(e) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

\begin{align*}
(1) \mathcal{L}(10\delta(t - \pi)), & \quad (2) \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)), & \quad (2) \mathcal{L}((t - \pi)\delta(t - \pi)).
\end{align*}

The sum of Dirac impulses in (2) is called an impulse train.
Solutions

Solution (a). The forward second shifting theorem applies.
(1) \( \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) \) where \( g(t) = 10 \) and \( a = \pi \). Then \( \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) = e^{-as} \mathcal{L}(g(t)|_{t=0}^{t+a}) = e^{-\pi s} \mathcal{L}(10|_{t=\pi}) = \frac{10e^{-\pi s}}{s} \).
(2) \( \mathcal{L}(\text{ramp}(t - 2\pi)) = \mathcal{L}((t - 2\pi)u(t - 2\pi)) = \mathcal{L}(tu(t)|_{t=2\pi}) = e^{-2\pi s} \mathcal{L}(t) = \frac{1}{s}e^{-2\pi s} \).
(3) \( \mathcal{L}(10 \text{pulse}(t,3,5)) = 10 \mathcal{L}(u(t - 3) - u(t - 5)) = \frac{10}{s}(e^{3s} - e^{-5s}) \).

Solution (b). Presence of an exponential \( e^{-as} \) signals \( u(t - a) \) in the answer, the main tool becoming the backward second shifting theorem.
(1) \( \mathcal{L}(f) = \frac{5e^{-3s}}{s} = e^{-3s}5 = e^{-3s} \mathcal{L}(5) = \mathcal{L}(5u(t)|_{t=0}^{t+3}) = \mathcal{L}(5u(t - 3)) \). Lerch implies \( f = 5u(t - 3) \).
(2) \( \mathcal{L}(f) = \frac{e^{-4s}}{s} = \frac{e^{-as}}{L}(t) \) where \( a = 4 \). Then \( \mathcal{L}(f) = \frac{e^{-as}}{L}(t) = \mathcal{L}(tu(t)|_{t=t-a}) = \mathcal{L}((t - 4)u(t - 4)) = \mathcal{L}(\text{ramp}(t - 4)) \). Lerch implies \( f = \text{ramp}(t - 4) \).
(3) \( \mathcal{L}(f) = e^{-2s}5 - e^{-3s}5 = \mathcal{L}(5u(t - 2)) - \mathcal{L}(5u(t - 3)) = \mathcal{L}(5 \text{pulse}(t,2,3)) \). Lerch implies \( f = 5 \text{pulse}(t,2,3) \).

Solution (c). The main result for Dirac unit impulse \( \delta \) is the equation
\[
\int_{0}^{\infty} g(t)\delta(t - a)dt = g(a),
\]
valid for \( g(t) \) continuous on \( 0 \leq t < \infty \). When \( g(t) = e^{-st} \), then the equation implies the Laplace formula \( \mathcal{L}(\delta(t - a)) = e^{-as} \).
(1) \( \mathcal{L}(10\delta(t - \pi)) = 10e^{-\pi s} \), by the displayed equation with \( g(t) = 10e^{-st} \), or by using linearity and the formula \( \mathcal{L}(\delta(t - a)) = e^{-as} \).
(2) \( \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)) = 5 \mathcal{L}(\delta(t - 1)) + 10 \mathcal{L}(\delta(t - 2)) + 15 \mathcal{L}(\delta(t - 3)) = 5e^{-s} + 10e^{-2s} + 15e^{-3s} \).
(3) \( \mathcal{L}((t - \pi)\delta(t - 2\pi)) = \int_{0}^{\infty} (t - \pi)e^{st}\delta(t - 2\pi)dt = (t - \pi)e^{-st}|_{t=2\pi} = \pi e^{-2\pi s} \), using \( g(t) = (t - \pi)e^{-st} \) and \( a = 2\pi \) in the equation.
Problem 3. Experiment to Find the Transfer Function $h(t)$

Consider a second order problem

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

which by Laplace theory has a particular solution defined as the convolution of the transfer function $h(t)$ with the input $f(t)$,

$$x_p(t) = \int_0^t f(w)h(t-w)dw.$$ 

Examined in this problem is another way to find $h(t)$, which is the system response to a Dirac unit impulse with zero data. Then $h(t)$ is the solution of

$$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.$$

The Problem. Assume $a, b, c$ are constants and define $g(t) = \int_0^t h(w)dw$.

(a) Show that $g(0) = g'(0) = 0$, which means $g$ has zero data.

(b) Let $u(t)$ be the unit step. Argue that $g$ is the solution of

$$ag''(t) + bg'(t) + cg(t) = u(t), \quad g(0) = g'(0) = 0.$$ 

The fundamental theorem of calculus says that $h(t) = g'(t)$. Therefore, to compute the transfer function $h(t)$, find the response $g(t)$ to the unit step with zero data, followed by computing the derivative $g'(t)$, which equals $h(t)$.

The experimental impact is important. Turning on a switch creates a unit step, generally easier than designing a hammer hit.

(c) Illustrate the method for finding the transfer function $h(t)$ in the special case

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

Solutions

(a) $g(0) = \int_0^0 h(w)dw = 0, \quad g'(0) = h'(0) = 0.$

(b) Let $u(t)$ be the unit step. Initial data was decided in part (a). The Laplace applied to $ag''(t) + bg'(t) + cg(t) = u(t)$ gives $(as^2 + bs + c) \mathcal{L}(g) = \mathcal{L}(u(t))$. Then $\mathcal{L}(g) = \mathcal{L}(h(t)) \mathcal{L}(u(t)) = \mathcal{L}(h(t)) \frac{1}{2} \mathcal{L} \left( \int_0^t h(r)dr \right)$ by the integral theorem. Lerch’s theorem then says $g(t) = \int_0^t h(r)dr.$

(c) For equation $x''(t) + 2x'(t) + 5x(t) = f(t)$ we replace $x(t)$ by $g(t)$ and $f(t)$ by the unit step $u(t)$, then solve $g''(t) + 2g'(t) + 5g(t) = u(t)$, obtaining $\mathcal{L}(g) = \frac{1}{s^2 + 2s + 5} = \mathcal{L} \left( \frac{1}{5} - \frac{1}{10} e^{-t} (2 \cos(2t) + \sin(2t)) \right)$. Then $g(t) = \frac{1}{5} - \frac{1}{10} e^{-t} (2 \cos(2t) + \sin(2t))$ and $h(t) = g'(t) = \frac{1}{2} e^{-t} \sin(2t)$. 