Quiz 6, Problem 1. Vertical Motion Seismoscope

The 1875 horizontal motion seismoscope of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.

A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is \( x(t) \).

The motion of the heavy mass \( m \) in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

\[
mx'' + cx' + kx = f(t)
\]

where \( f(t) \) is the vertical ground force due to the earthquake. In terms of the vertical ground motion \( u(t) \), we write via Newton’s second law the force equation \( f(t) = -mu''(t) \) (compare to falling body \(-mg\)). The final model for the motion of the mass is then

\[
\begin{cases}
x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\
c \frac{m}{m} = 2\beta\Omega_0, \quad k \frac{m}{m} = \Omega_0^2, \\
x(t) = \text{center of mass position measured from equilibrium}, \\
u(t) = \text{vertical ground motion due to the earthquake}.
\end{cases}
\]

Terms seismoscope, seismograph, seismometer refer to the device in the figure. Some observations:

- Slow ground movement means \( x' \approx 0 \) and \( x'' \approx 0 \), then \([1]\) implies \( \Omega_0^2 x(t) = -u''(t) \). The seismometer records ground acceleration.

- Fast ground movement means \( x \approx 0 \) and \( x' \approx 0 \), then \([1]\) implies \( x''(t) = -u''(t) \). The seismometer records ground displacement.

A release test begins by starting a vibration with \( u \) identically zero. Two successive maxima \( (t_1, x_1), (t_2, x_2) \) are recorded. This experiment determines constants \( \beta, \Omega_0 \).

The objective of \([1]\) is to determine \( u(t) \), by knowing \( x(t) \) from the seismograph.

The Problem.

Assume the seismograph trace can be modeled at time \( t = 0 \) (a time after the earthquake struck) by \( x(t) = 10\cos(3t) \). Assume a release test determined \( 2\beta\Omega_0 = 16 \) and \( \Omega_0^2 = 80 \). Explain how to find a formula for the ground motion \( u(t) \), then provide details for the answer \( u(t) = \frac{710}{9} \cos(3t) - \frac{160}{3} \sin(3t) \) (assume integration constants are zero).
The **Branch Current Method** can be used to find a $3 \times 3$ linear system for the **branch currents** $I_1, I_2, I_3$.

\[
\begin{align*}
I_1 - I_2 - I_3 &= 0 \quad \text{KCL, upper node} \\
3I_1 + 2I_2 &= 18 \quad \text{KVL, left loop} \\
2I_2 - 2I_3 &= 5 \quad \text{KVL, right loop}
\end{align*}
\]

Symbol **KCL** means *Kirchhoff’s Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff’s Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

(a) Solve the equations to find the currents $I_1, I_2, I_3$ in the figure.

(b) Compute the voltage drops across resistors $R_1, R_2, R_3$. Answer: $\frac{93}{8}$, $\frac{51}{8}$, $\frac{11}{8}$ volts.

(c) Replace the 5 volt battery by a 4 volt battery. Solve the system again, and report the new currents and voltage drops.

**References.** Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:
http://www.allaboutcircuits.com/

Course slides on Electric Circuits:

Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:
http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html
The Problem. Suppose $E = \sin(40t)$, $L = 1$ H, $R = 50$ Ω and $C = 0.01$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $I'' + 50I' + 100I = 40\cos(40t)$.

(b) Find the reactance $S = \omega L - \frac{1}{\omega C}$, where $\omega = 40$ is the input frequency, the natural frequency of $E = \sin(40t)$ and $E' = 40\cos(40t)$. Then find the impedance $Z = \sqrt{S^2 + R^2}$.

(c) The steady-state current is $I(t) = A\cos(40t) + B\sin(40t)$ for some constants $A, B$. Substitute $I = A\cos(40t) + B\sin(40t)$ into the current model (a) and solve for $A, B$.

Answers: $A = -\frac{6}{625}, B = \frac{8}{625}$.

(d) Write the answer in (c) in phase-amplitude form $I = I_0\sin(40t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the time lag $\delta/\omega$.

Answers: $I_0 = 0.016, \delta = \arctan(0.75), \delta/\omega = 0.0160875$.

References

Course slides on Electric Circuits:

Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.