Quiz 4

Quiz 4, Problem 1. Picard–Lindelöf Theorem and Spring-Mass Models

Picard-Lindelöf Theorem. Let \( \vec{f}(x, \vec{y}) \) be defined for \(|x - x_0| \leq h, \|\vec{y} - \vec{y}_0\| \leq k\), with \( \vec{f} \) and \( \frac{\partial \vec{f}}{\partial \vec{y}} \) continuous. Then for some constant \( H, 0 < H < h \), the problem

\[
\begin{align*}
\vec{y}'(x) &= \vec{f}(x, \vec{y}(x)), \\
\vec{y}(x_0) &= \vec{y}_0
\end{align*}
\]

has a unique solution \( \vec{y}(x) \) defined on the smaller interval \(|x - x_0| < H\).

The Problem. The second order problem

\[
\begin{align*}
u'' + 2u' + 17u &= 100, \\
u(0) &= 1, \\
u'(0) &= -1
\end{align*}
\]

is a spring-mass model with damping and constant external force. The variables are time \( x \) in seconds and elongation \( u(x) \) in meters, measured from equilibrium. Coefficients in the equation represent mass \( m = 1 \) kg, a viscous damping constant \( c = 2 \), Hooke’s constant \( k = 17 \) and external force \( F(x) = 100 \).

Convert the scalar initial value problem into a vector problem, to which Picard’s vector theorem applies, by supplying details for the parts below.

(a) The conversion uses the position-velocity substitution \( y_1 = u(x), y_2 = u'(x) \), where \( y_1, y_2 \) are the invented components of vector \( \vec{y} \). Then the initial data \( u(0) = 1, u'(0) = -1 \) converts to the vector initial data

\[
\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

(b) Differentiate the equations \( y_1 = u(x), y_2 = u'(x) \) in order to find the scalar system of two differential equations, known as a dynamical system:

\[
y'_1 = y_2, \quad y'_2 = -17y_1 - 2y_2 + 100.
\]

(c) The derivative of vector function \( \vec{y}(x) \) is written \( \vec{y}'(x) \) or \( \frac{\partial \vec{y}}{\partial x}(x) \). It is obtained by componentwise differentiation: \( \vec{y}'(x) = \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} \). The vector differential equation model of scalar system (1) is

\[
\begin{align*}
\vec{y}'(x) &= \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\
\vec{y}(0) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{align*}
\]

(d) System (2) fits the hypothesis of Picard’s theorem, using symbols

\[
\begin{align*}
\vec{f}(x, \vec{y}) &= \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\
\vec{y}_0 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{align*}
\]

The components of vector function \( \vec{f} \) are continuously differentiable in variables \( x, y_1, y_2 \), therefore \( \vec{f} \) and \( \frac{\partial \vec{f}}{\partial \vec{y}} \) are continuous.
Quiz4 Problem 2. The velocity of a crossbow bolt launched upward from the ground was determined from a video and a speed gun to complete the following table.

<table>
<thead>
<tr>
<th>Time $t$ in seconds</th>
<th>Velocity $v(t)$ in ft/sec</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>60</td>
<td>Ground</td>
</tr>
<tr>
<td>1.7</td>
<td>0</td>
<td>Maximum</td>
</tr>
<tr>
<td>3.5</td>
<td>-52</td>
<td>Near Ground Impact</td>
</tr>
</tbody>
</table>

(a) The bolt velocity can be approximated by a quadratic polynomial

$$v(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients $a, b, c$. Then solve for the coefficients.

(b) Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.11$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 60$ with $\rho = 0.11$.

(d) The error between $v(t)$ and $w(t)$ can be measured. Is the drag coefficient value $\rho = 0.11$ reasonable?

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

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Quiz4 Extra Credit Problem 3. Consider the system of differential equations

$$
\begin{align*}
    x'_1 &= -\frac{1}{5}x_1 + \frac{1}{7}x_3, \\
    x'_2 &= \frac{1}{5}x_1 - \frac{1}{5}x_2, \\
    x'_3 &= \frac{1}{5}x_2 - \frac{1}{7}x_3,
\end{align*}
$$

for the amounts $x_1, x_2, x_3$ of salt in recirculating brine tanks, as in the figure:

Recirculating Brine Tanks A, B, C  
The volumes are 50, 30, 70 for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x'_1 = x'_2 = x'_3 = 0$ and then solving for the symbols $x_1, x_2, x_3$. 

(a) Solve the corresponding linear system of algebraic equations for answers $x_1, x_2, x_3$.

(b) The total amount of salt is uniformly distributed in the tanks in ratio 5 : 3 : 7. Explain this mathematically from the answer in (a).

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.