Background. Switches and Impulses

Laplace’s method solves differential equations. It is the premier method for solving equations containing switches or impulses.

**Unit Step** Define \( u(t - a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases} \). It is a switch, turned on at \( t = a \).

**Ramp** Define \( \text{ramp}(t - a) = (t - a)u(t - a) = \begin{cases} t - a & t \geq a, \\ 0 & t < a. \end{cases} \), whose graph shape is a continuous ramp at 45-degree incline starting at \( t = a \).

**Unit Pulse** Define \( \text{pulse}(t,a,b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b). \) The switch is ON at time \( t = a \) and then OFF at time \( t = b \).

**Impulse of a Force**

Define the impulse of an applied force \( F(t) \) on time interval \( a \leq t \leq b \) by the equation

\[
\text{Impulse of } F = \int_{a}^{b} F(t)dt = \left( \int_{a}^{b} F(t)dt \right) / (b - a) = \text{Average Force } \times \text{Duration Time}.
\]

**Dirac Unit Impulse**

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the impulse of the force is deemed important, and not its magnitude nor duration.

Define the Dirac Unit Impulse by the equation \( \delta(t - a) = \frac{du}{dt}(t - a) \), where \( u(t - a) \) is the unit step. Symbol \( \delta \) makes sense only under an integral sign, and the integral in question must be a generalized Riemann-Stieltjes integral (definition pending), with new evaluation rules. Symbol \( \delta \) is an abbreviation like etc or e.g., because it abbreviates a paragraph of descriptive text.

- Symbol \( M\delta(t - a) \) represents an ideal impulse of magnitude \( M \) at time \( t = a \). Value \( M \) is the change in momentum, but \( M\delta(t - a) \) contains no detail about the applied force or the duration. A common force approximation for a hammer hit of very small duration \( 2h \) and impulse \( M \) is Dirac’s approximation

\[
F_h(t) = \frac{M}{2h} \text{pulse}(t, a - h, a + h).
\]

- The fundamental equation is \( \int_{-\infty}^{\infty} F(x)\delta(x - a)dx = F(a) \). Symbol \( \delta(t - a) \) is not manipulated as an ordinary function, but regarded as \( du(t - a)/dt \) in a Riemann-Stieltjes integral.

**THEOREM** (Second Shifting Theorem). Let \( f(t) \) and \( g(t) \) be piecewise continuous and of exponential order. Then for \( a \geq 0 \),

**Forward table**

\[
\mathcal{L}(f(t-a)u(t-a)) = e^{-as} \mathcal{L}(f(t)) \\
\mathcal{L}(g(t)u(t-a)) = e^{-as} \mathcal{L} \left( g(t) \right)_{t:=t+a}
\]

**Backward table**

\[
e^{-as} \mathcal{L}(f(t)) = \mathcal{L} \left( f(t-a)u(t-a) \right) \\
e^{-as} \mathcal{L}(f(t)) = \mathcal{L} \left( f(t)u(t) \right)_{t:=t-a}
\]
Problem 1. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for terms involving the unit step, ramp and pulse, in these special cases:

(1) \( \mathcal{L}((t - 1)u(t - 1)) \)  \( \mathcal{L}(e^t \textbf{ramp}(t - 2)) \),  \( \mathcal{L}(5 \textbf{pulse}(t, 2, 4)) \).

(b) Backward table. Find \( f(t) \) in the following special cases.

(1) \( \mathcal{L}(f) = \frac{e^{-2s}}{s} \)  \( \mathcal{L}(f) = \frac{e^{-s}}{(s + 1)^2} \)  \( \mathcal{L}(f) = e^{-s} \frac{3}{s} - e^{-2s} \frac{3}{s} \).
Problem 2. Solve the following Dirac impulse problem.

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

\[ (1) \mathcal{L}(2\delta(t - 5)), \quad (2) \mathcal{L}(2\delta(t - 1) + 5\delta(t - 3)), \quad (3) \mathcal{L}(e^t\delta(t - 2)). \]

The sum of Dirac impulses in (2) is called an impulse train. The numbers 2 and 5 represent the applied impulse at times 1 and 3, respectively.
Reference: The Riemann-Stieltjes Integral

Definition
The Riemann-Stieltjes integral of a real-valued function $f$ of a real variable with respect to a real monotone non-decreasing function $g$ is denoted by

$$
\int_{a}^{b} f(x) \, dg(x)
$$

and defined to be the limit, as the mesh of the partition

$$
P = \{a = x_0 < x_1 < \cdots < x_n = b\}
$$

of the interval $[a, b]$ approaches zero, of the approximating Riemann-Stieltjes sum

$$
S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))
$$

where $c_i$ is in the $i$-th subinterval $[x_i, x_{i+1}]$. The two functions $f$ and $g$ are respectively called the integrand and the integrator.

The limit is a number $A$, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\text{mesh}(P) = \max_{0 \leq i < n} (x_{i+1} - x_i) < \delta$, and for every choice of points $c_i$ in $[x_i, x_{i+1}]$,

$$
|S(P, f, g) - A| < \varepsilon.
$$