9 Fourier Series Methods

9.1 Periodic Functions and Trigonometric Series

As motivation for the subject of Fourier series, we consider the differential equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = f(t), \tag{1}$$

which models the behavior of a mass-and-spring system with natural (circular) frequency ω_0, moving under the influence of an external force of magnitude $f(t)$ per unit mass. As we saw in Section 3.6, a particular solution of Eq. (1) can easily be found by the method of undetermined coefficients if $f(t)$ is a simple harmonic function—a sine or cosine function. For instance, the equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = A\cos\omega t \tag{2}$$

with $\omega^2 \neq \omega_0^2$ has the particular solution

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t, \tag{3}$$

which is readily found by beginning with the trial solution $x_p(t) = a\cos\omega t$.

Now suppose, more generally, that the force function $f(t)$ in Eq. (1) is a linear combination of simple harmonic functions. Then, on the basis of Eq. (3) and the analogous formula with sine in place of cosine, we can apply the principle of superposition to construct a particular solution of Eq. (1). For example, consider the equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \sum_{n=1}^{N} A_n \cos\omega_n t, \tag{4}$$

in which ω_0^2 is equal to none of the ω_n^2. Equation (4) has the particular solution

$$x_p(t) = \sum_{n=1}^{N} \frac{A_n}{\omega_0^2 - \omega_n^2} \cos \omega_n t, \tag{5}$$

obtained by adding the solutions given in Eq. (3) corresponding to the N terms on the right-hand side in Eq. (4).

Mechanical (and electrical) systems often involve periodic forcing functions that are not (simply) finite linear combinations of sines and cosines. Nevertheless, as we will soon see, any reasonably nice periodic function $f(t)$ has a representation as an *infinite series* of trigonometric terms. This fact opens the way toward solving Eq. (1) by superposition of trigonometric "building blocks," with the finite sum in Eq. (5) replaced with an infinite series.

DEFINITION Periodic Function

The function $f(t)$ defined for all t is said to be **periodic** provided that there exists a positive number p such that

$$f(t + p) = f(t) \tag{6}$$

for all t. The number p is then called a **period** of the function f.

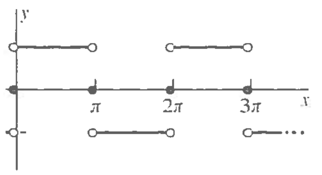
Note that the period of a periodic function is not unique; for example, if p is a period of $f(t)$, then so are the numbers $2p$, $3p$, and so on. Indeed, every positive number is a period of any constant function.

If there exists a smallest positive number P such that $f(t)$ is periodic with period P, then we call P *the* period of f. For instance, the period of the functions $g(t) = \cos nt$ and $h(t) = \sin nt$ (where n is a positive integer) is $2\pi/n$ because

$$\cos n\left(t + \frac{2\pi}{n}\right) = \cos(nt + 2\pi) = \cos nt$$

and (7)

$$\sin n\left(t + \frac{2\pi}{n}\right) = \sin(nt + 2\pi) = \sin nt.$$



URE 9.1.1. A square-wave ion.

Moreover, 2π itself is a period of the functions $g(t)$ and $h(t)$. Ordinarily we will have no need to refer to the smallest period of a function $f(t)$ and will simply say that $f(t)$ has period p if p is any period of $f(t)$.

In Section 7.5 we saw several examples of piecewise continuous periodic functions. For instance, the square-wave function having the graph shown in Fig. 9.1.1 has period 2π.

Because $g(t) = \cos nt$ and $h(t) = \sin nt$ each have period 2π, any linear combination of sines and cosines of integral multiples of t, such as

$$f(t) = 3 + \cos t - \sin t + 5\cos 2t + 17\sin 3t,$$

has period 2π. But every such linear combination is continuous, so the square-wave function cannot be expressed in such manner. In his celebrated treatise *The Analytic Theory of Heat* (1822), the French scientist Joseph Fourier (1768–1830) made the remarkable assertion that every function $f(t)$ with period 2π can be represented by an *infinite* trigonometric series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{8}$$

(The reason for writing $\frac{1}{2}a_0$ rather than a_0 here will appear shortly—when we see that a single formula for a_n thereby includes the case $n = 0$ as well as $n > 0$.) We will see in Section 9.2 that under rather mild restrictions on the function $f(t)$, this is so! An infinite series of the form in (8) is called a ***Fourier series***, and the representation of functions by Fourier series is one of the most widely used techniques in applied mathematics, especially for the solution of partial differential equations (see Sections 9.5 through 9.7).

Fourier Series of Period 2π Functions

In this section we will confine our attention to functions of period 2π. We want to determine what the coefficients in the Fourier series in (8) must be if it is to converge to a given function $f(t)$ of period 2π. For this purpose we need the following integrals, in which m and n denote positive integers (Problems 27 through 29):

$$\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases} \tag{9}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases} \tag{10}$$

$$\int_{-\pi}^{\pi} \cos mt \sin nt \, dt = 0 \quad \text{for all } m \text{ and } n. \tag{11}$$

These formulas imply that the functions $\cos nt$ and $\sin nt$ for $n = 1, 2, 3, \ldots$ constitute a *mutually orthogonal* set of functions on the interval $[-\pi, \pi]$. Two real-valued functions $u(t)$ and $v(t)$ are said to be **orthogonal** on the interval $[a, b]$ provided that

$$\int_a^b u(t)v(t)\,dt = 0. \tag{12}$$

(The reason for the word "orthogonal" here is a certain interpretation of functions as vectors with infinitely many values or "components," in which the integral of the product of two functions plays the same role as the dot product of two ordinary vectors; recall that $\mathbf{u}\cdot\mathbf{v} = 0$ if and only if the two vectors are orthogonal.)

Suppose now that the piecewise continuous function $f(t)$ of period 2π has a Fourier series representation

➤ $$f(t)=\frac{a_0}{2}+\sum_{m=1}^{\infty}(a_m \cos mt + b_m \sin mt), \tag{13}$$

in the sense that the infinite series on the right converges to the value $f(t)$ for every t. We assume in addition that, when the infinite series in Eq. (13) is multiplied by any continuous function, the resulting series can be integrated term by term. Then the result of termwise integration of Eq. (13) itself from $t = -\pi$ to $t = \pi$ is

$$\int_{-\pi}^{\pi} f(t)\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi} 1\,dt + \sum_{m=1}^{\infty}\left(a_m \int_{-\pi}^{\pi} \cos mt\,dt\right) + \sum_{m=1}^{\infty}\left(b_m \int_{-\pi}^{\pi} \sin mt\,dt\right) = \pi a_0$$

because all the trigonometric integrals vanish. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt. \tag{14}$$

If we first multiply each side in Eq. (13) by $\cos nt$ and then integrate termwise, the result is

$$\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nt\,dt + \sum_{m=1}^{\infty}\left(a_m \int_{-\pi}^{\pi} \cos mt\,\cos nt\,dt\right) + \sum_{m=1}^{\infty}\left(b_m \int_{-\pi}^{\pi} \sin mt\,\cos nt\,dt\right);$$

it then follows from Eq. (11) that

$$\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \sum_{m=1}^{\infty} a_m \left(\int_{-\pi}^{\pi} \cos mt\,\cos nt\,dt\right). \tag{15}$$

But Eq. (9) says that—of all the integrals (for $m = 1, 2, 3, \ldots$) on the right-hand side in (15)—only the one for which $m = n$ is nonzero. It follows that

$$\int_{-\pi}^{\pi} f(t)\cos nt\,dt = a_n \int_{-\pi}^{\pi} \cos^2 nt\,dt = \pi a_n,$$

so the value of the coefficient a_n is

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt. \tag{16}$$

Note that with $n = 0$, the formula in (16) reduces to Eq. (14); this explains why we denote the constant term in the original Fourier series by $\frac{1}{2}a_0$ (rather than simply a_0). If we multiply each side in Eq. (13) by $\sin nt$ and then integrate termwise, we find in a similar way that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \tag{17}$$

(Problem 31). In short, we have found that *if* the series in (13) converges to $f(t)$ and *if* the termwise integrations carried out here are valid, *then* the coefficients in the series must have the values given in Eqs. (16) and (17). This motivates us to *define* the Fourier series of a periodic function by means of these formulas, whether or not the resulting series converges to the function (or even converges at all).

DEFINITION Fourier Series and Fourier Coefficients

Let $f(t)$ be a piecewise continuous function of period 2π that is defined for all t. Then the **Fourier series** of $f(t)$ is the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \tag{18}$$

where the **Fourier coefficients** a_n and b_n are defined by means of the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \tag{16}$$

for $n = 0, 1, 2, 3, \ldots$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \tag{17}$$

for $n = 1, 2, 3, \ldots$.

You may recall that the Taylor series of a function sometimes fails to converge everywhere to the function whence it came. It is still more common that the Fourier series of a given function sometimes fails to converge to its actual values at certain points in the domain of the function. We will therefore write

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \tag{19}$$

not using an equals sign between the function and its Fourier series until we have discussed convergence of Fourier series in Section 9.2.

Suppose that the piecewise continuous function $f(t)$ as given initially is defined only on the interval $[-\pi, \pi]$, and assume that $f(-\pi) = f(\pi)$. Then we can extend f so that its domain includes all real numbers by means of the periodicity condition $f(t + 2\pi) = f(t)$ for all t. We continue to denote this extension of the original function by f, and note that it automatically has period 2π. Its graph looks the same on every interval of the form

$$(2n-1)\pi \leqq t \leqq (2n+1)\pi$$

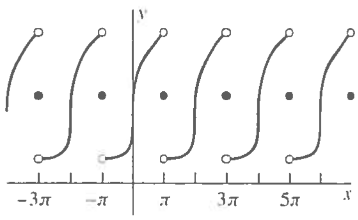


FIGURE 9.1.2. Extending a function to produce a periodic function.

where n is an integer (Fig. 9.1.2). For instance, the square-wave function of Fig. 9.1.1 can be described as the period 2π function such that

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0; \\ +1 & \text{if } 0 < t < \pi; \\ 0 & \text{if } t = -\pi, 0, \text{ or } \pi. \end{cases} \tag{20}$$

Thus the square-wave function is the period 2π function defined on one full period by means of Eq. (20).

We need to consider Fourier series of piecewise continuous functions because many functions that appear in applications are only piecewise continuous, not continuous. Note that the integrals in Eqs. (16) and (17) exist if $f(t)$ is piecewise continuous, so every piecewise continuous function has a Fourier series.

Example 1 Find the Fourier series of the square-wave function defined in Eq. (20).

Solution It is always a good idea to calculate a_0 separately, using Eq. (14). Thus

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(+1)\,dt$$

$$= \frac{1}{\pi}(-\pi) + \frac{1}{\pi}(\pi) = 0.$$

We split the first integral into two integrals because $f(t)$ is defined by different formulas on the intervals $(-\pi, 0)$ and $(0, \pi)$; the values of $f(t)$ at the endpoints of these intervals do not affect the values of the integrals.

Equation (16) yields (for $n > 0$)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-\cos nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\cos nt\,dt$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\sin nt\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{1}{n}\sin nt\right]_{0}^{\pi} = 0.$$

And Eq. (17) yields

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin nt\,dt$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cos nt\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{1}{n}\cos nt\right]_{0}^{\pi}$$

$$= \frac{2}{n\pi}(1 - \cos n\pi) = \frac{2}{n\pi}[1 - (-1)^n].$$

Thus $a_n = 0$ for all $n \geqq 0$, and

$$b_n = \begin{cases} \dfrac{4}{n\pi} & \text{for } n \text{ odd;} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

The last result follows because $\cos(-n\pi) = \cos(n\pi) = (-1)^n$. With these values of the Fourier coefficients, we obtain the Fourier series

$$f(t) \sim \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n} = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right). \tag{21}$$

Here we have introduced the useful abbreviation

$$\sum_{n\text{ odd}} \quad \text{for} \quad \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}$$

—for example,

$$\sum_{n\text{ odd}}\frac{1}{n} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots.$$

Figure 9.1.3 shows the graphs of several of the partial sums

$$S_N(t) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{\sin(2n-1)t}{2n-1}$$

of the Fourier series in (21). Note that as t approaches a discontinuity of $f(t)$ from either side, the value of $S_n(t)$ tends to overshoot the limiting value of $f(t)$—either $+1$ or -1 in this case. This behavior of a Fourier series near a point of discontinuity of its function is typical and is known as **Gibbs's phenomenon**. ■

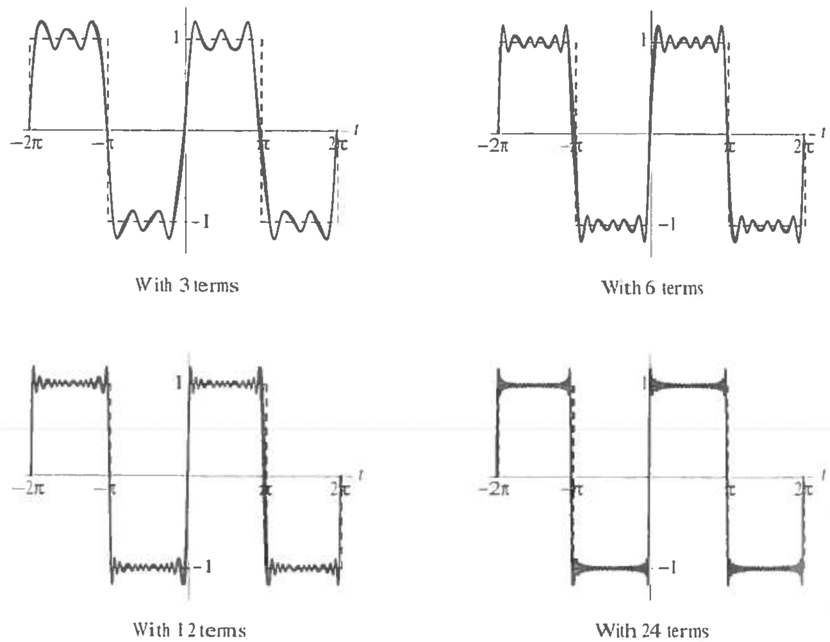


FIGURE 9.1.3. Graphs of partial sums of the Fourier series of the square-wave function (Example 1) with $N = 3, 6, 12$, and 24 terms.

The following integral formulas, easily derived by integration by parts, are useful in computing Fourier series of polynomial functions:

$$\int u \cos u \, du = \cos u + u \sin u + C; \tag{22}$$

$$\int u \sin u \, du = \sin u - u \cos u + C; \tag{23}$$

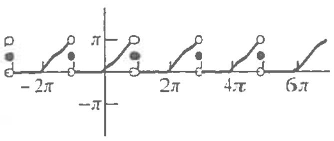
$$\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du; \tag{24}$$

$$\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du. \tag{25}$$

Example 2 Find the Fourier series of the period 2π function that is defined in one period to be

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leqq 0; \\ t & \text{if } 0 \leqq t < \pi; \\ \dfrac{\pi}{2} & \text{if } t = \pm\pi. \end{cases} \tag{26}$$

The graph of f is shown in Fig. 9.1.4.



GURE 9.1.4. The periodic
nction of Example 2.

Solution The values of $f(\pm\pi)$ are irrelevant because they have no effect on the values of the integrals that yield the Fourier coefficients. Because $f(t) \equiv 0$ on the interval $(-\pi, 0)$, each integral from $t = -\pi$ to $t = \pi$ may be replaced with an integral from $t = 0$ to $t = \pi$. Equations (14), (16), and (17) therefore give

$$a_0 = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{1}{\pi}\left[\frac{1}{2}t^2\right]_0^{\pi} = \frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt = \frac{1}{n^2\pi}\int_0^{n\pi} u\cos u\,du \qquad \left(u = nt,\ t = \frac{u}{n}\right)$$

$$= \frac{1}{n^2\pi}\Big[\cos u + u\sin u\Big]_0^{n\pi} \qquad \text{(by Eq. (22))}$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1].$$

Consequently, $a_n = 0$ if n is even and $n \geqq 2$;

$$a_n = -\frac{2}{n^2\pi} \qquad \text{if } n \text{ is odd.}$$

Next,

$$b_n = \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt = \frac{1}{n^2\pi}\int_0^{n\pi} u\sin u\,du$$

$$= \frac{1}{n^2\pi}\Big[\sin u - u\cos u\Big]_0^{n\pi} \qquad \text{(by Eq. (20))}$$

$$= -\frac{1}{n}\cos n\pi.$$

Thus

$$b_n = \frac{(-1)^{n+1}}{n} \qquad \text{for all } n \geqq 1.$$

Therefore, the Fourier series of $f(t)$ is

$$f(t) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nt}{n}. \tag{27}$$

■

If $f(t)$ is a function of period 2π, it is readily verified (Problem 30) that

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{a}^{a+2\pi} f(t)\,dt \tag{28}$$

for all a. That is, the integral of $f(t)$ over one interval of length 2π is equal to its integral over any other such interval. In case $f(t)$ is given explicitly on the interval $[0, 2\pi]$ rather than on $[-\pi, \pi]$, it may be more convenient to compute its Fourier coefficients as

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt \tag{29a}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt. \tag{29b}$$

9.1 Problems

In Problems 1 through 10, sketch the graph of the function f defined for all t by the given formula, and determine whether it is periodic. If so, find its smallest period.

1. $f(t) = \sin 3t$
2. $f(t) = \cos 2\pi t$
3. $f(t) = \cos \dfrac{3t}{2}$
4. $f(t) = \sin \dfrac{\pi t}{3}$
5. $f(t) = \tan t$
6. $f(t) = \cot 2\pi t$
7. $f(t) = \cosh 3t$
8. $f(t) = \sinh \pi t$
9. $f(t) = |\sin t|$
10. $f(t) = \cos^2 3t$

In Problems 11 through 26, the values of a period 2π function $f(t)$ in one full period are given. Sketch several periods of its graph and find its Fourier series.

11. $f(t) \equiv 1,\ -\pi \leqq t \leqq \pi$
12. $f(t) = \begin{cases} +3, & -\pi < t \leqq 0; \\ -3, & 0 < t \leqq \pi \end{cases}$
13. $f(t) = \begin{cases} 0, & -\pi < t \leqq 0; \\ 1, & 0 < t \leqq \pi \end{cases}$
14. $f(t) = \begin{cases} 3, & -\pi < t \leqq 0; \\ -2, & 0 < t \leqq \pi \end{cases}$
15. $f(t) = t,\ -\pi < t \leqq \pi$
16. $f(t) = t,\ 0 < t < 2\pi$
17. $f(t) = |t|,\ -\pi \leqq t \leqq \pi$
18. $f(t) = \begin{cases} \pi + t, & -\pi < t \leqq 0; \\ \pi - t, & 0 < t \leqq \pi \end{cases}$
19. $f(t) = \begin{cases} \pi + t, & -\pi \leqq t < 0; \\ 0, & 0 \leqq t \leqq \pi \end{cases}$
20. $f(t) = \begin{cases} 0, & -\pi \leqq t < -\pi/2; \\ 1, & -\pi/2 \leqq t \leqq \pi/2; \\ 0, & \pi/2 < t \leqq \pi \end{cases}$
21. $f(t) = t^2,\ -\pi \leqq t \leqq \pi$
22. $f(t) = t^2,\ 0 \leqq t < 2\pi$
23. $f(t) = \begin{cases} 0, & -\pi \leqq t \leqq 0; \\ t^2, & 0 \leqq t < \pi \end{cases}$
24. $f(t) = |\sin t|,\ -\pi \leqq t \leqq \pi$
25. $f(t) = \cos^2 2t,\ -\pi \leqq t \leqq \pi$
26. $f(t) = \begin{cases} 0, & -\pi \leqq t \leqq 0; \\ \sin t, & 0 \leqq t \leqq \pi \end{cases}$
27. Verify Eq. (9). (*Suggestion:* Use the trigonometric identity
$$\cos A\cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)].)$$
28. Verify Eq. (10).
29. Verify Eq. (11).
30. Let $f(t)$ be a piecewise continuous function with period P. (a) Suppose that $0 \leqq a < P$. Substitute $u = t - P$ to show that
$$\int_P^{a+P} f(t)\,dt = \int_0^a f(t)\,dt.$$
Conclude that
$$\int_a^{a+P} f(t)\,dt = \int_0^P f(t)\,dt.$$
(b) Given A, choose n so that $A = nP + a$ with $0 \leqq a < P$. Then substitute $v = t - nP$ to show that
$$\int_A^{A+P} f(t)\,dt = \int_a^{a+P} f(t)\,dt = \int_0^P f(t)\,dt.$$
31. Multiply each side in Eq. (13) by $\sin nt$ and then integrate term by term to derive Eq. (17).

9.2 General Fourier Series and Convergence

In Section 9.1 we defined the Fourier series of a periodic function of period 2π. Now let $f(t)$ be a function that is piecewise continuous for all t and has arbitrary period $P > 0$. We write

$$P = 2L, \tag{1}$$

so L is the **half-period** of the function f. Let us define the function g as follows:

$$g(u) = f\left(\frac{Lu}{\pi}\right) \tag{2}$$

for all u. Then

$$g(u+2\pi) = f\left(\frac{Lu}{\pi}+2L\right) = f\left(\frac{Lu}{\pi}\right) = g(u),$$

and hence $g(u)$ is also periodic and has period 2π. Consequently, g has the Fourier series

$$g(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nu + b_n \sin nu) \tag{3}$$

with Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu\, du \tag{4a}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin nu\, du. \tag{4b}$$

If we now write

$$t = \frac{Lu}{\pi}, \quad u = \frac{\pi t}{L}, \quad f(t) = g(u), \tag{5}$$

then

$$f(t) = g\left(\frac{\pi t}{L}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \frac{n\pi t}{L}\right), \tag{6}$$

and then substitution of (5) in (4) yields

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu\, du \qquad \left(u = \frac{\pi t}{L},\ du = \frac{\pi}{L}\,dt\right) \\ &= \frac{1}{L}\int_{-L}^{L} g\left(\frac{\pi t}{L}\right)\cos\frac{n\pi t}{L}\,dt. \end{aligned}$$

Therefore,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt; \tag{7}$$

similarly,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt. \tag{8}$$

This computation motivates the following definition of the Fourier series of a periodic function of period $2L$.

DEFINITION Fourier Series and Fourier Coefficients

Let $f(t)$ be a piecewise continuous function of period $2L$ that is defined for all t. Then the **Fourier series** of $f(t)$ is the series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right), \tag{6}$$

where the **Fourier coefficients** $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ are defined to be

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \tag{7}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt. \tag{8}$$

With $n = 0$, Eq. (7) takes the simple form

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt, \tag{9}$$

which demonstrates that the constant term $\frac{1}{2}a_0$ in the Fourier series of f is simply the average value of $f(t)$ on the interval $[-L, L]$.

As a consequence of Problem 30 of Section 9.1, we may evaluate the integrals in (7) and (8) over any other interval of length $2L$. For instance, if $f(t)$ is given by a single formula for $0 < t < 2L$, it may be more convenient to compute the integrals

$$a_n = \frac{1}{L}\int_{0}^{2L} f(t)\cos\frac{n\pi t}{L}\,dt \tag{10a}$$

and

$$b_n = \frac{1}{L}\int_{0}^{2L} f(t)\sin\frac{n\pi t}{L}\,dt. \tag{10b}$$

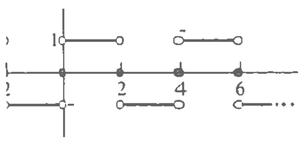
Example 1 Figure 9.2.1 shows the graph of a square-wave function with period 4. Find its Fourier series.

Solution Here, $L = 2$; also, $f(t) = -1$ if $-2 < t < 0$, while $f(t) = 1$ if $0 < t < 2$. Hence Eqs. (7), (8), and (9) yield

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\int_{-2}^{0}(-1)\,dt + \frac{1}{2}\int_{0}^{2}(+1)\,dt = 0,$$

$$a_n = \frac{1}{2}\int_{-2}^{0}(-1)\cos\frac{n\pi t}{2}\,dt + \frac{1}{2}\int_{0}^{2}(+1)\cos\frac{n\pi t}{2}\,dt$$

$$= \frac{1}{2}\left[-\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right]_{0}^{2} = 0,$$



RE 9.2.1. The :-wave of Example 1.

and

$$b_n = \frac{1}{2}\int_{-2}^{0}(-1)\sin\frac{n\pi t}{2}\,dt + \frac{1}{2}\int_{0}^{2}(+1)\sin\frac{n\pi t}{2}\,dt$$

$$+\frac{1}{2}\left[\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_{-2}^{0} + \frac{1}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_{0}^{2}$$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus the Fourier series is

$$f(t) \sim \frac{4}{\pi}\sum_{n\ \text{odd}} \frac{1}{n}\sin\frac{n\pi t}{2} \tag{11a}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} + \frac{1}{5}\sin\frac{5\pi t}{2} + \cdots\right). \tag{11b}$$

■

The Convergence Theorem

We want to impose conditions on the periodic function f that are enough to guarantee that its Fourier series actually converges to $f(t)$ at least at those values of t at which f is continuous. Recall that the function f is said to be *piecewise continuous* on the interval $[a, b]$ provided that there is a finite partition of $[a, b]$ with endpoints

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

such that

1. f is continuous on each open interval $t_{i-1} < t < t_i$; and
2. At each endpoint t_i of such a subinterval the limit of $f(t)$, as t approaches t_i from within the subinterval, exists and is finite.

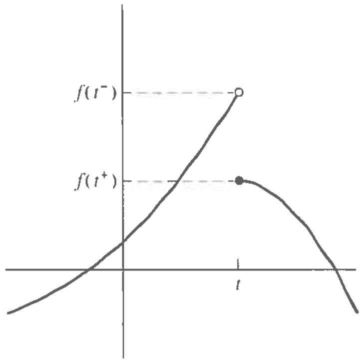


FIGURE 9.2.2. A finite jump discontinuity.

The function f is called *piecewise continuous* for all t if it is piecewise continuous on every bounded interval. It follows that a piecewise continuous function is continuous except possibly at isolated points, and that at each such point of discontinuity, the one-sided limits

$$f(t+) = \lim_{u \to t^+} f(u) \quad \text{and} \quad f(t-) = \lim_{u \to t^-} f(u) \tag{12}$$

both exist and are finite. Thus a piecewise continuous function has only isolated "finite jump" discontinuities like the one shown in Fig. 9.2.2.

The square-wave and sawtooth functions that we saw in Chapter 7 are typical examples of periodic piecewise continuous functions. The function $f(t) = \tan t$ is a periodic function (of period π) that is not piecewise continuous because it has infinite discontinuities. The function $g(t) = \sin(1/t)$ is not piecewise continuous on $[-1, 1]$ because its one-sided limits at $t = 0$ do not exist. The function

$$h(t) = \begin{cases} t & \text{if } t = \dfrac{1}{n} \quad (n \text{ an integer}), \\ 0 & \text{otherwise} \end{cases}$$

on $[-1, 1]$ has one-sided limits everywhere, but is not piecewise continuous because its discontinuities are not isolated—it has the infinite sequence $\{1/n\}_1^\infty$ of discontinuities; a piecewise continuous function can have only finitely many discontinuities in any bounded interval.

Note that a piecewise continuous function need not be defined at its isolated points of discontinuity. Alternatively, it can be defined arbitrarily at such points. For instance, the square wave function f of Fig. 9.2.1 is piecewise continuous no matter what its values might be at the points $\dots, -4, -2, 0, 2, 4, 6, \dots$ at which it is discontinuous. Its derivative f' is also piecewise continuous; $f'(t) = 0$ unless t is an even integer, in which case $f'(t)$ is undefined.

The piecewise continuous function f is said to be **piecewise smooth** provided that its derivative f' is piecewise continuous. Theorem 1 (next) tells us that the Fourier series of a piecewise smooth function converges everywhere. More general Fourier convergence theorems—with weaker hypotheses on the periodic function f—are known. But the hypothesis that f is piecewise smooth is easy to check and is satisfied by most functions encountered in practical applications. A proof of the following theorem may be found in G. P. Tolstov, *Fourier Series* (New York: Dover, 1976).

THEOREM 1 Convergence of Fourier Series

Suppose that the periodic function f is piecewise smooth. Then its Fourier series in (6) converges

(a) to the value $f(t)$ at each point where f is continuous, and

(b) to the value $\frac{1}{2}[f(t+) + f(t-)]$ at each point where f is discontinuous.

Note that $\frac{1}{2}[f(t+) + f(t-)]$ is the *average* of the right-hand and left-hand limits of f at the point t. If f is continuous at t, then $f(t) = f(t+) = f(t-)$, so

$$f(t) = \frac{f(t+) + f(t-)}{2}. \tag{13}$$

Hence Theorem 1 could be rephrased as follows: The Fourier series of a piecewise smooth function f converges for *every* t to the average value in (13). For this reason it is customary to write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right), \tag{14}$$

with the understanding that the piecewise smooth function f has been redefined (if necessary) at each of its points of discontinuity in order to satisfy the average value condition in (13).

Example 1
Continued

Figure 9.2.1 shows us at a glance that if t_0 is an even integer, then

$$\lim_{t\to t_0^+} f(t) = +1 \quad \text{and} \quad \lim_{t\to t_0^-} f(t) = -1.$$

Hence

$$\frac{f(t_0+) + f(t_0-)}{2} = 0.$$

Note that, in accord with Theorem 1, the Fourier series of $f(t)$ in (11) clearly converges to zero if n is an even integer (because $\sin n\pi = 0$). ■

Example 2

Let $f(t)$ be a function of period 2 with $f(t) = t^2$ if $0 < t < 2$. We define $f(t)$ for t an even integer by the average value condition in (13); consequently, $f(t) = 2$ if t is an even integer. The graph of the function f appears in Fig. 9.2.3. Find its Fourier series.

Solution

Here $L = 1$, and it is most convenient to integrate from $t = 0$ to $t = 2$. Then

$$a_0 = \frac{1}{1}\int_0^2 t^2\,dt = \left[\frac{1}{3}t^3\right]_0^2 = \frac{8}{3}.$$

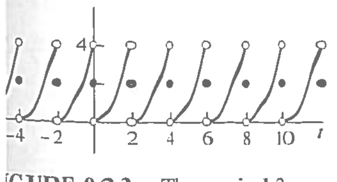


FIGURE 9.2.3. The period 2 function of Example 2.

With the aid of the integral formulas in Eqs. (22) through (25) of Section 9.1, we obtain

$$\begin{aligned}
a_n &= \int_0^2 t^2 \cos n\pi t\,dt \\
&= \frac{1}{n^3\pi^3}\int_0^{2n\pi} u^2 \cos u\,du \qquad \left(u = n\pi t,\ t = \frac{u}{n\pi}\right) \\
&= \frac{1}{n^3\pi^3}\Big[u^2 \sin u - 2\sin u + 2u\cos u\Big]_0^{2n\pi} = \frac{4}{n^2\pi^2};
\end{aligned}$$

$$\begin{aligned}
b_n &= \int_0^2 t^2 \sin n\pi t\,dt = \frac{1}{n^3\pi^3}\int_0^{2n\pi} u^2 \sin u\,du \\
&= \frac{1}{n^3\pi^3}\Big[-u^2 \cos u + 2\cos u + 2u \sin u\Big]_0^{2n\pi} = -\frac{4}{n\pi}.
\end{aligned}$$

Hence the Fourier series of f is

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}, \tag{15}$$

and Theorem 1 assures us that this series converges to $f(t)$ for all t. ■

We can draw some interesting consequences from the Fourier series in (15). If we substitute $t = 0$ on each side, we find that

$$f(0) = 2 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

On solving for the series, we obtain the lovely summation

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \tag{16}$$

that was discovered by Euler. If we substitute $t = 1$ in Eq. (15), we get

$$f(1) = 1 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

which yields

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}. \tag{17}$$

If we add the series in Eqs. (16) and (17) and then divide by 2, the "even" terms cancel and the result is

$$\sum_{n \text{ odd}}\frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}. \tag{18}$$

9.2 Problems

In Problems 1 through 14, the values of a periodic function $f(t)$ in one full period are given; at each discontinuity the value of $f(t)$ is that given by the average value condition in (13). Sketch the graph of f and find its Fourier series.

1. $f(t) = \begin{cases} -2, & -3 < t < 0; \\ 2, & 0 < t < 3 \end{cases}$

2. $f(t) = \begin{cases} 0, & -5 < t < 0; \\ 1, & 0 < t < 5 \end{cases}$

3. $f(t) = \begin{cases} 2, & -2\pi < t < 0; \\ -1, & 0 < t < 2\pi \end{cases}$

4. $f(t) = t, \; -2 < t < 2$

5. $f(t) = t, \; -2\pi < t < 2\pi$

6. $f(t) = t, \; 0 < t < 3$

7. $f(t) = |t|, \; -1 < t < 1$

8. $f(t) = \begin{cases} 0, & 0 < t < 1; \\ 1, & 1 < t < 2; \\ 0, & 2 < t < 3 \end{cases}$

9. $f(t) = t^2, \; -1 < t < 1$

10. $f(t) = \begin{cases} 0, & -2 < t < 0; \\ t^2, & 0 < t < 2 \end{cases}$

11. $f(t) = \cos\frac{\pi t}{2}, \; -1 < t < 1$

$f(t) = \sin \pi t,\ 0 < t < 1$

$$f(t) = \begin{cases} 0, & -1 < t < 0; \\ \sin \pi t, & 0 < t < 1 \end{cases}$$

$$f(t) = \begin{cases} 0, & -2\pi < t < 0; \\ \sin t, & 0 < t < 2\pi \end{cases}$$

(a) Suppose that f is a function of period 2π with $f(t) = t^2$ for $0 < t < 2\pi$. Show that

$$f(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nt}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

and sketch the graph of f, indicating the value at each discontinuity. **(b)** Deduce the series summations in Eqs. (16) and (17) from the Fourier series in part (a).

(a) Suppose that f is a function of period 2 such that $f(t) = 0$ if $-1 < t < 0$ and $f(t) = t$ if $0 < t < 1$. Show that

$$f(t) = \frac{1}{4} - \frac{2}{\pi^2} \sum_{n \text{ odd}} \frac{\cos n\pi t}{n^2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi t}{n}$$

and sketch the graph of f, indicating the value at each discontinuity. **(b)** Deduce the series summation in Eq. (18) from the Fourier series in part (a).

(a) Suppose that f is a function of period 2 with $f(t) = t$ for $0 < t < 2$. Show that

$$f(t) = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}$$

and sketch the graph of f, indicating the value at each discontinuity. **(b)** Substitute an appropriate value of t to deduce **Leibniz's series**

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

ve the Fourier series listed in Problems 18 through 21, and h the period 2π function to which each series converges.

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n} = \frac{\pi - t}{2} \qquad (0 < t < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nt}{n} = \frac{t}{2} \qquad (-\pi < t < \pi)$$

$$\sum_{n=1}^{\infty} \frac{\cos nt}{n^2} = \frac{3t^2 - 6\pi t + 2\pi^2}{12} \qquad (0 < t < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nt}{n^2} = \frac{\pi^2 - 3t^2}{12} \qquad (-\pi < t < \pi)$$

Suppose that $p(t)$ is a polynomial of degree n. Show by repeated integration by parts that

$$\int p(t) g(t)\, dt = p(t)G_1(t) - p'(t)G_2(t) + p''(t)G_3(t) - \cdots + (-1)^n p^{(n)}(t)G_{n+1}(t)$$

where $G_k(t)$ denotes the kth iterated antiderivative $G_k(t) = (D^{-1})^k g(t)$. This formula is useful in computing Fourier coefficients of polynomials.

23. Apply the integral formula of Problem 22 to show that

$$\int t^4 \cos t\, dt = t^4 \sin t + 4t^3 \cos t - 12t^2 \sin t - 24t \cos t + 24 \sin t + C$$

and that

$$\int t^4 \sin t\, dt = -t^4 \cos t + 4t^3 \sin t + 12t^2 \cos t - 24t \sin t - 24 \cos t + C.$$

24. **(a)** Show that for $0 < t < 2\pi$,

$$t^4 = \frac{16\pi^4}{5} + 16 \sum_{n=1}^{\infty} \left(\frac{2\pi^2}{n^2} - \frac{3}{n^4} \right) \cos nt + 16\pi \sum_{n=1}^{\infty} \left(\frac{3}{n^3} - \frac{\pi^2}{n} \right) \sin nt$$

and sketch the graph of f, indicating the value at each discontinuity. **(b)** From the Fourier series in part (a), deduce the summations

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720},$$

and

$$\sum_{n \text{ odd}} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

25. **(a)** Find the Fourier series of the period 2π function f with $f(t) = t^3$ if $-\pi < t < \pi$. **(b)** Use the series of part (a) to derive the summation

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

and sketch the graph of f, indicating the value at each discontinuity. **(c)** *Attempt* to evaluate the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

by substituting an appropriate value of t in the Fourier series of part (a). Is your attempt successful? Explain. *Remark*: If you succeed in expressing the sum of this inverse-cube series in terms of familiar numbers—for instance, as a rational multiple of π^3 similar to Euler's sum in part (a)—you will win great fame for yourself, for many have tried without success over the last two centuries since Euler. Indeed, it was not until 1979 that the sum of the inverse-cube series was proved to be an irrational number (as long suspected).

9.2 Application Computer Algebra Calculation of Fourier Coefficients

A computer algebra system can greatly ease the burden of calculation of the Fourier coefficients of a given function $f(t)$. In the case of a function that is defined "piecewise," we must take care to "split" the integral according to the different intervals of definition of the function. We illustrate the method by deriving the Fourier series of the period 2π square-wave function defined on $(-\pi, \pi)$ by

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0, \\ +1 & \text{if } 0 < t < \pi. \end{cases} \tag{1}$$

In this case the function is defined by different formulas on two different intervals, so each Fourier coefficient integral from $-\pi$ to π must be calculated as the sum of two integrals:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (+1)\cos nt\,dt, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (+1)\sin nt\,dt. \end{aligned} \tag{2}$$

We can define the coefficients in (2) as functions of n by the *Maple* commands

```
a := n -> (1/Pi)*(int(-cos(n*t), t=-Pi..0) +
                    int(+cos(n*t), t=0..Pi)):
b := n -> (1/Pi)*(int(-sin(n*t), t=-Pi..0) +
                    int(+sin(n*t), t=0..Pi)):
```

or by the *Mathematica* commands

```
a[n_] := (1/Pi)*(Integrate[-Cos[n*t], {t, -Pi, 0}] +
                    Integrate[+Cos[n*t], {t, 0, Pi}])
b[n_] := (1/Pi)*(Integrate[-Sin[n*t], {t, -Pi, 0}] +
                    Integrate[+Sin[n*t], {t, 0, Pi}])
```

Because the function $f(t)$ in Eq. (1) is odd, we naturally find that $a_n \equiv 0$. Hence the *Maple* commands

```
fourierSum := sum(b(n)*sin(n*t), n=1..9);
plot(fourierSum, t=-2*Pi..4*Pi);
```

or the *Mathematica* commands

```
fourierSum = Sum[b[n]*Sin[n*t], {n,1,9}]
Plot[fourierSum, {t, -2*Pi, 4*Pi}];
```

yield the partial sum

$$\sum_{n=1}^{9} b_n \sin nt = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \frac{\sin 9t}{9}\right)$$

and generate a graph like one of those in Fig. 9.1.3. The corresponding MATLAB commands are entirely analogous and can be found in the applications manual that accompanies this text.

To practice the symbolic derivation of Fourier series in this manner, you can begin by verifying the Fourier series calculated manually in Examples 1 and 2 of this section. Then Problems 1 through 21 are fair game. Finally, the period 2π triangular wave and trapezoidal wave functions illustrated in Figs. 9.2.4 and 9.2.5 have especially interesting Fourier series that we invite you to discover for yourself.

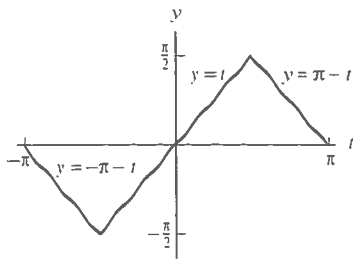


FIGURE 9.2.4. The triangular wave.

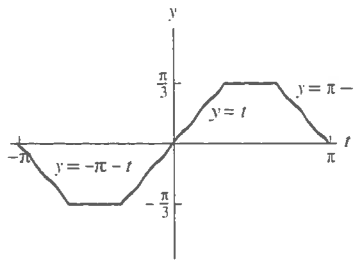


FIGURE 9.2.5. The trapezoidal wave.

9.3 Fourier Sine and Cosine Series

Certain properties of functions are reflected prominently in their Fourier series. The function f defined for all t is said to be **even** if

$$f(-t) = f(t) \tag{1}$$

for all t; f is **odd** if

$$f(-t) = -f(t) \tag{2}$$

for all t. The first condition implies that the graph of $y = f(t)$ is symmetric with respect to the y-axis, whereas the condition in (2) implies that the graph of an odd function is symmetric with respect to the origin (see Fig. 9.3.1). The functions $f(t) = t^{2n}$ (with n an integer) and $g(t) = \cos t$ are even functions, whereas the functions $f(t) = t^{2n+1}$ and $g(t) = \sin t$ are odd. We will see that the Fourier series of an even periodic function has only cosine terms and that the Fourier series of an odd periodic function has only sine terms.

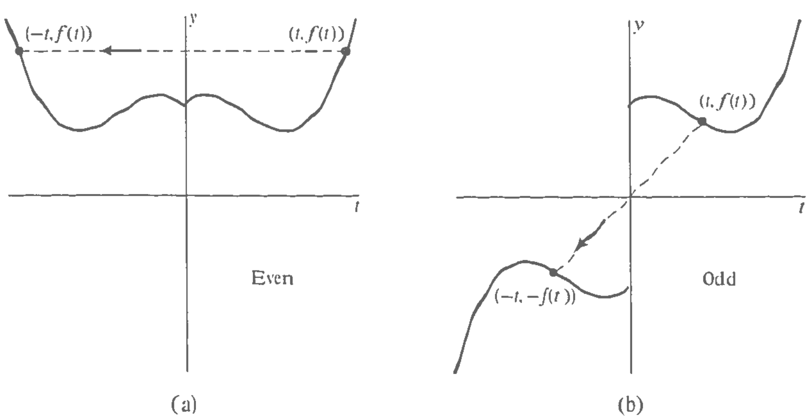


FIGURE 9.3.1. (a) An even function; (b) an odd function.

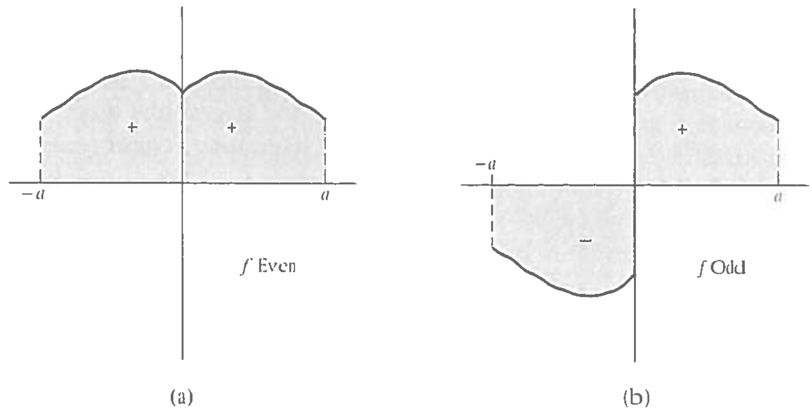


FIGURE 9.3.2. Area under the graph of (a) an even function and (b) an odd.

Addition and cancellation of areas as indicated in Fig. 9.3.2 reminds us of the following basic facts about integrals of even and odd functions over an interval $[-a, a]$ that is symmetric around the origin.

$$\text{If } f \text{ is even:} \quad \int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt. \tag{3}$$

$$\text{If } f \text{ is odd:} \quad \int_{-a}^{a} f(t)\,dt = 0. \tag{4}$$

These facts are easy to verify analytically (Problem 17).

It follows immediately from Eqs. (1) and (2) that the product of two even functions is even, as is the product of two odd functions; the product of an even function and an odd function is odd. In particular, if $f(t)$ is an *even* periodic function of period $2L$, then $f(t)\cos(n\pi t/L)$ is even, whereas $f(t)\sin(n\pi t/L)$ is odd, because the cosine function is even and the sine function is odd. When we compute the Fourier coefficients of f, we therefore get

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{L}\int_{0}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \tag{5a}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = 0 \tag{5b}$$

because of (3) and (4). Hence the Fourier series of the *even* function f of period $2L$ has only *cosine* terms:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} \qquad (f \text{ even}) \tag{6}$$

with the values of a_n given by Eq. (5a). If $f(t)$ is *odd*, then $f(t)\cos(n\pi t/L)$ is odd, whereas $f(t)\sin(n\pi t/L)$ is even, so

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = 0 \tag{7a}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = \frac{2}{L}\int_{0}^{L} f(t)\sin\frac{n\pi t}{L}\,dt. \tag{7b}$$

Hence the Fourier series of the *odd* function f of period $2L$ has only *sine* terms:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L} \qquad (f \text{ odd}) \tag{8}$$

with the coefficients b_n given in Eq. (7b).

Even and Odd Extensions

In all our earlier discussion and examples, we began with a periodic function defined *for all* t; the Fourier series of such a function is uniquely determined by the Fourier coefficient formulas. In many practical situations, however, we begin with a function f defined only on an interval of the form $0 < t < L$, and we want to represent its values on this interval by a Fourier series of period $2L$. The first step is the necessary extension of f to the interval $-L < t < 0$. Granted this, we may extend f to the entire real line by the periodicity condition $f(t + 2L) = f(t)$ (and use the average value property should any discontinuities arise). But *how* we define f for $-L < t < 0$ is our choice, and the Fourier series representation for $f(t)$ on $(0, L)$ that we obtain will depend on that choice. Specifically, different choices of the extension of f to the interval $(-L, 0)$ will yield different Fourier series that converge to the same function $f(t)$ in the original interval $0 < t < L$, but converge to the different extensions of f on the interval $-L < t < 0$.

In practice, given $f(t)$ defined for $0 < t < L$, we generally make one of two natural choices—we extend f in such a way as to obtain either an *even function* or an *odd function* on the whole real line. The **even period $2L$ extension** of f is the function f_E defined as

$$f_E(t) = \begin{cases} f(t) & \text{if } \ 0 < t < L, \\ f(-t) & \text{if } -L < t < 0 \end{cases} \tag{9}$$

and by $f_E(t + 2L) = f_E(t)$ for all t. The **odd period $2L$ extension** of f is the function f_O defined as

$$f_O(t) = \begin{cases} f(t) & \text{if } \ 0 < t < L, \\ -f(-t) & \text{if } -L < t < 0 \end{cases} \tag{10}$$

and by $f_O(t + 2L) = f_O(t)$ for all t. The values of f_E or f_O for t an integral multiple of L can be defined in any convenient way we wish, because these isolated values cannot affect the Fourier series of the extensions we get. As suggested by Fig. 9.3.1, it frequently suffices simply to visualize the graph of f_E on $(-L, 0)$ as the reflection in the vertical axis of the original graph of f on $(0, L)$, and the graph of f_O on $(-L, 0)$ as the reflection in the origin of the original graph.

For instance, if $f(t) = 2t - t^2$ on the interval $0 < t < 2$ (so $L = 2$), then (9) and (10) yield

$$f_E(t) = 2(-t) - (-t)^2 = -2t - t^2$$

and

$$f_O(t) = -\left[2(-t) - (-t)^2\right] = 2t + t^2$$

for the values of these two extensions on the interval $-2 < t < 0$. The graphs of the corresponding two periodic extensions of f are shown in Fig. 9.3.3.

The Fourier series of the even extension f_E of the function f, given by Eqs. (5) and (6), will contain only cosine terms and is called the *Fourier cosine series* of the original function f. The Fourier series of the odd extension f_O, given by Eqs. (7) and (8), will contain only sine terms and is called the *Fourier sine series* of f.

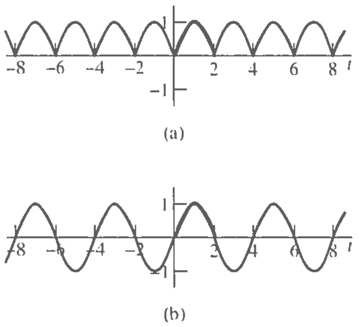


FIGURE 9.3.3. (a) The period 4 even extension of $f(t) = 2t - t^2$ for $0 < t < 2$. (b) Graph of the period 4 odd extension of $f(t) = 2t - t^2$ for $0 < t < 2$.

DEFINITION Fourier Cosine and Sine Series

Suppose that the function $f(t)$ is piecewise continuous on the interval $[0, L]$. Then the **Fourier cosine series** of f is the series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} \tag{11}$$

with

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt. \tag{12}$$

The **Fourier sine series** of f is the series

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L} \tag{13}$$

with

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt. \tag{14}$$

Assuming that f is piecewise smooth and satisfies the average value condition $f(t) = \frac{1}{2}[f(t+) + f(t-)]$ at each of its isolated discontinuities, Theorem 1 of Section 9.2 implies that each of the two series in (11) and (13) converges to $f(t)$ for all t in the interval $0 < t < L$. Outside this interval, the cosine series in (11) converges to the even period $2L$ extension of f, whereas the sine series in (13) converges to the odd period $2L$ extension of f. In many cases of interest we have no concern with the values of f outside the original interval $(0, L)$, and therefore the choice between (11) and (12) or (13) and (14) is determined by whether we prefer to represent $f(t)$ in the interval $(0, L)$ by a cosine series or a sine series. (See Example 2 for a situation that dictates our choice between a Fourier cosine series and a Fourier sine series to represent a given function.)

Example 1 Suppose that $f(t) = t$ for $0 < t < L$. Find both the Fourier cosine series and the Fourier sine series for f.

Solution Equation (12) gives

$$a_0 = \frac{2}{L}\int_0^L t\,dt = \frac{2}{L}\left[\frac{1}{2}t^2\right]_0^L = L$$

and

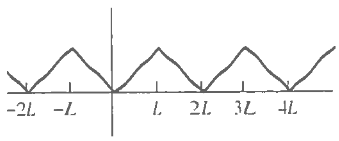
$$a_n = \frac{2}{L}\int_0^L t\cos\frac{n\pi t}{L}\,dt = \frac{2L}{n^2\pi^2}\int_0^{n\pi} u\cos u\,du$$

$$= \frac{2L}{n^2\pi^2}\Big[u\sin u + \cos u\Big]_0^{n\pi} = \begin{cases} -\dfrac{4L}{n^2\pi^2} & \text{for } n \text{ odd;} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Thus the Fourier cosine series of f is

$$t = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi t}{L} + \frac{1}{3^2}\cos\frac{3\pi t}{L} + \frac{1}{5^2}\cos\frac{5\pi t}{L} + \cdots\right) \tag{15}$$

for $0 < t < L$. Next, Eq. (14) gives

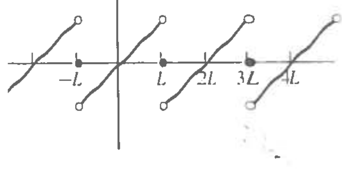


GURE 9.3.4. The even period , extension of f.

$$b_n = \frac{2}{L}\int_0^L t\sin\frac{n\pi t}{L}\,dt = \frac{2L}{n^2\pi^2}\int_0^{n\pi} u\sin u\,du$$

$$= \frac{2L}{n^2\pi^2}\Big[-u\cos u + \sin u\Big]_0^{n\pi} = \frac{2L}{n\pi}(-1)^{n+1}.$$

Thus the Fourier sine series of f is



GURE 9.3.5. The odd period , extension of f.

$$t = \frac{2L}{\pi}\left(\sin\frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - \cdots\right) \tag{16}$$

for $0 < t < L$. The series in Eq. (15) converges to the even period $2L$ extension of f shown in Fig. 9.3.4; the series in Eq. (16) converges to the odd period $2L$ extension shown in Fig. 9.3.5. ■

Termwise Differentiation of Fourier Series

In this and in subsequent sections, we want to consider Fourier series as possible solutions of differential equations. In order to substitute a Fourier series for the unknown dependent variable in a differential equation to check whether it is a solution, we first need to differentiate the series in order to compute the derivatives that appear in the equation. Care is required here; term-by-term differentiation of an infinite series of variable terms is not always valid. Theorem 1 gives sufficient conditions for the validity of termwise differentiation of a Fourier series.

THEOREM 1 Termwise Differentiation of Fourier Series

Suppose that the function f is continuous for all t, periodic with period $2L$, and that its derivative f' is piecewise smooth for all t. Then the Fourier series of f' is the series

$$f'(t) = \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L}a_n\sin\frac{n\pi t}{L} + \frac{n\pi}{L}b_n\cos\frac{n\pi t}{L}\right) \tag{17}$$

obtained by termwise differentiation of the Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right). \tag{18}$$

***Proof*:** The point of the theorem is that the differentiated series in Eq. (17) actually converges to $f'(t)$ (with the usual proviso about average values). But because f' is periodic and piecewise smooth, we know from Theorem 1 of Section 9.2 that the Fourier series of f' converges to $f'(t)$:

$$f'(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\alpha_n \cos\frac{n\pi t}{L} + \beta_n \sin\frac{n\pi t}{L}\right). \tag{19}$$

In order to prove Theorem 1, it therefore suffices to show that the series in Eqs. (17) and (19) are identical. We will do so under the additional hypothesis that f' is continuous everywhere. Then

$$\alpha_0 = \frac{1}{L}\int_{-L}^{L} f'(t)\,dt = \frac{1}{L}\Big[f(t)\Big]_{-L}^{L} = 0$$

because $f(L) = f(-L)$ by periodicity, and

$$\begin{aligned}\alpha_n &= \frac{1}{L}\int_{-L}^{L} f'(t)\cos\frac{n\pi t}{L}\,dt\\ &= \frac{1}{L}\left[f(t)\cos\frac{n\pi t}{L}\right]_{-L}^{L} + \frac{n\pi}{L}\cdot\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt\end{aligned}$$

—integration by parts. It follows that

$$\alpha_n = \frac{n\pi}{L}b_n.$$

Similarly, we find that

$$\beta_n = -\frac{n\pi}{L}a_n,$$

and therefore the series in Eqs. (17) and (19) are, indeed, identical. ▲

Whereas the assumption that the derivative f' is continuous is merely a convenience—the proof of Theorem 1 can be strengthened to allow isolated discontinuities in f'—it is important to note that the conclusion of Theorem 1 generally fails when f itself is discontinuous. For example, consider the Fourier series

$$t = \frac{2L}{\pi}\left(\sin\frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - \cdots\right), \tag{16}$$

$-L < t < L$, of the discontinuous sawtooth function having the graph shown in Fig. 9.3.5. All the hypotheses of Theorem 1 are satisfied apart from the continuity of f, and f has only isolated jump discontinuities. But the series

$$2\left(\cos\frac{\pi t}{L} - \cos\frac{2\pi t}{L} + \cos\frac{3\pi t}{L} - \cdots\right) \tag{20}$$

obtained by differentiating the series in Eq. (16) term by term diverges (for instance, when $t = 0$ and when $t = L$), and therefore termwise differentiation of the series in Eq. (16) is not valid.

By contrast, consider the (continuous) triangular wave function $f(t)$ having the graph shown in Fig. 9.3.4, with $f(t) = |t|$ for $-L < t < L$. This function satisfies all the hypotheses of Theorem 1, so its Fourier series

$$f(t) = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi t}{L} + \frac{1}{3^2}\cos\frac{3\pi t}{L} + \frac{1}{5^2}\cos\frac{5\pi t}{L} + \cdots\right) \tag{15}$$

can be differentiated termwise. The result is

$$f'(t) = \frac{4}{\pi}\left(\sin\frac{\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} + \frac{1}{5}\sin\frac{5\pi t}{L} + \cdots\right), \tag{21}$$

which is the Fourier series of the period $2L$ square wave function that takes the value -1 for $-L < t < 0$ and $+1$ for $0 < t < L$.

Fourier Series Solutions of Differential Equations

In the remainder of this chapter and in Chapter 10, we will frequently need to solve endpoint value problems of the general form

$$ax'' + bx' + cx = f(t) \quad (0 < t < L); \tag{22}$$

$$x(0) = x(L) = 0, \tag{23}$$

where the function $f(t)$ is given. Of course, we might consider applying the techniques of Chapter 3, solving the problem by

1. First finding the general solution $x_c = c_1x_1 + c_2x_2$ of the associated homogeneous differential equation;
2. Then finding a single particular solution x_p of the nonhomogeneous equation in (22); and
3. Finally, determining the constants c_1 and c_2 so that $x = x_c + x_p$ satisfies the endpoint conditions in (23).

In many problems, however, the following Fourier series method is more convenient and more useful. We first extend the definition of the function $f(t)$ to the interval $-L < t < 0$ in an appropriate way, and then to the entire real line by the periodicity conditions $f(t+2L) = f(t)$. Then the function f, if piecewise smooth, has a Fourier series

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi t}{L} + B_n\sin\frac{n\pi t}{L}\right), \tag{24}$$

which has coefficients $\{A_n\}$ and $\{B_n\}$ that we can and do compute. We then assume that the differential equation in (22) has a solution $x(t)$ with a Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right) \tag{25}$$

that may validly be differentiated twice termwise. We attempt to determine the coefficients in Eq. (25) by first substituting the series in Eqs. (24) and (25) into the differential equation in (22) and then equating coefficients of like terms—much as in the ordinary method of undetermined coefficients (Section 3.5), except that now we have infinitely many coefficients to determine. If this procedure is carried out in such a way that the resulting series in Eq. (25) also satisfies the endpoint conditions in (23), then we have a "formal Fourier series solution" of the original endpoint value problem; that is, a solution subject to verification of the assumed termwise differentiability. Example 2 illustrates this process.

Example 2 Find a formal Fourier series solution of the endpoint value problem

$$x'' + 4x = 4t, \tag{26}$$

$$x(0) = x(1) = 0. \tag{27}$$

Solution Here $f(t) = 4t$ for $0 < t < 1$. A crucial first step—which we did not make explicit in the preceding outline—is to choose a periodic extension $f(t)$ so that each term in its Fourier series satisfies the endpoint conditions in (27). For this purpose we choose the odd period 2 extension, because each term of the form $\sin n\pi t$ satisfies (27). Then from the series in Eq. (16) with $L = 1$, we get the Fourier series

$$4t = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi t \tag{28}$$

for $0 < t < 1$. We therefore anticipate a sine series solution

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t, \tag{29}$$

noting that any such series will satisfy the endpoint conditions in (27). When we substitute the series in (28) and (29) in Eq. (26), the result is

$$\sum_{n=1}^{\infty}(-n^2\pi^2 + 4)b_n \sin n\pi t = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi t. \tag{30}$$

We next equate coefficients of like terms in Eq. (30). This yields

$$b_n = \frac{8\cdot(-1)^{n+1}}{n\pi(4 - n^2\pi^2)},$$

so our formal Fourier series solution is

$$x(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi t}{n(4 - n^2\pi^2)}. \tag{31}$$

In Problem 16 we ask you to derive the exact solution

$$x(t) = t - \frac{\sin 2t}{\sin 2} \qquad (0 \leqq t \leqq 1), \tag{32}$$

and to verify that (31) is the Fourier series of the odd period 2 extension of this solution.

The dashed curve in Fig. 9.3.6 was plotted by summing 10 terms of the Fourier series in (31). The solid curve for $0 \leqq t \leqq 2$ is the graph of the exact solution in (32). ∎

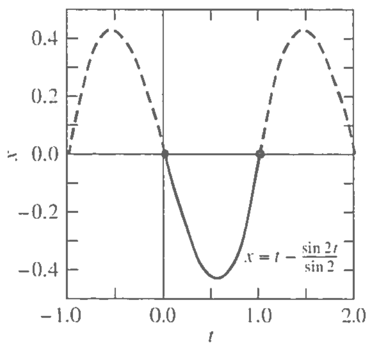


FIGURE 9.3.6. Graph of the solution in Example 2.

Termwise Integration of Fourier Series

Theorem 2 guarantees that the Fourier series of a piecewise continuous periodic function can always be integrated term by term, whether or not it converges! A proof is outlined in Problem 25.

THEOREM 2 Termwise Integration of Fourier Series

Suppose that f is a piecewise continuous periodic function with period $2L$ and Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right), \tag{33}$$

which may not converge. Then

$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} \frac{L}{n\pi} \left[a_n \sin \frac{n\pi t}{L} - b_n \left(\cos \frac{n\pi t}{L} - 1 \right) \right], \tag{34}$$

with the series on the right-hand side convergent for all t. Note that the series in Eq. (34) is the result of term-by-term integration of the series in (33), but if $a_0 \neq 0$ it is not a Fourier series because of its linear initial term $\frac{1}{2}a_0 t$.

Example 3 Let us attempt to verify the conclusion of Theorem 2 in the case that $f(t)$ is the period 2π function such that

$$f(t) = \begin{cases} -1, & -\pi < t < 0; \\ +1, & 0 < t < \pi. \end{cases} \tag{35}$$

By Example 1 of Section 9.1, the Fourier series of f is

$$f(t) = \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots \right). \tag{36}$$

Theorem 2 then implies that

$$\begin{aligned} F(t) &= \int_0^t f(s)\,ds \\ &= \int_0^t \frac{4}{\pi} \left(\sin s + \frac{1}{3} \sin 3s + \frac{1}{5} \sin 5s + \cdots \right) ds \\ &= \frac{4}{\pi} \left[(1 - \cos t) + \frac{1}{3^2}(1 - \cos 3t) + \frac{1}{5^2}(1 - \cos 5t) + \cdots \right]. \end{aligned}$$

Thus

$$\begin{aligned} F(t) &= \frac{4}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) \\ &\quad - \frac{4}{\pi} \left(\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right). \end{aligned} \tag{37}$$

On the other hand, direct integration of (35) yields

$$F(t) = \int_0^t f(s)\,ds = |t| = \begin{cases} -t, & -\pi < t < 0, \\ t, & 0 < t < \pi. \end{cases}$$

We know from Example 1 in this section (with $L = \pi$) that

$$|t| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right). \tag{38}$$

We also know from Eq. (18) of Section 9.2 that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8},$$

so it follows that the two series in Eqs. (37) and (38) are indeed identical. ■

9.3 Problems

In Problems 1 through 10, a function $f(t)$ defined on an interval $0 < t < L$ is given. Find the Fourier cosine and sine series of f and sketch the graphs of the two extensions of f to which these two series converge.

1. $f(t) = 1, 0 < t < \pi$

2. $f(t) = 1 - t, 0 < t < 1$

3. $f(t) = 1 - t, 0 < t < 2$

4. $f(t) = \begin{cases} t, & 0 < t \leqq 1; \\ 2 - t, & 1 \leqq t < 2 \end{cases}$

5. $f(t) = \begin{cases} 0, & 0 < t < 1; \\ 1, & 1 < t < 2; \\ 0, & 2 < t < 3 \end{cases}$

6. $f(t) = t^2, 0 < t < \pi$

7. $f(t) = t(\pi - t), 0 < t < \pi$

8. $f(t) = t - t^2, 0 < t < 1$

9. $f(t) = \sin t, 0 < t < \pi$

10. $f(t) = \begin{cases} \sin t, & 0 < t \leqq \pi \\ 0, & \pi \leqq t < 2\pi \end{cases}$

Find formal Fourier series solutions of the endpoint value problems in Problems 11 through 14.

11. $x'' + 2x = 1, x(0) = x(\pi) = 0$

12. $x'' - 4x = 1, x(0) = x(\pi) = 0$

13. $x'' + x = t, x(0) = x(1) = 0$

14. $x'' + 2x = t, x(0) = x(2) = 0$

15. Find a formal Fourier series solution of the endpoint value problem

$$x'' + 2x = t, \quad x'(0) = x'(\pi) = 0.$$

(*Suggestion*: Use a Fourier cosine series in which each term satisfies the endpoint conditions.)

16. **(a)** Derive the solution $x(t) = t - (\sin 2t)/(\sin 2)$ of the endpoint value problem

$$x'' + 4x = 4t, \quad x(0) = x(1) = 0.$$

(b) Show that the series in Eq. (31) is the Fourier sine series of the solution in part (a).

17. **(a)** Suppose that f is an even function. Show that

$$\int_{-a}^{0} f(t)\,dt = \int_0^a f(t)\,dt.$$

(b) Suppose that f is an odd function. Show that

$$\int_{-a}^{0} f(t)\,dt = -\int_0^a f(t)\,dt.$$

18. By Example 2 of Section 9.2, the Fourier series of the period 2 function f with $f(t) = t^2$ for $0 < t < 2$ is

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}.$$

Show that the termwise derivative of this series does not converge to $f'(t)$.

19. Begin with the Fourier series

$$t = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt, \quad -\pi < t < \pi,$$

and integrate termwise three times in succession to obtain the series

$$\frac{1}{24}t^4 = \frac{\pi^2 t^2}{12} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos nt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}.$$

ubstitute $t = \pi/2$ and $t = \pi$ in the series of Problem 19
) obtain the summations

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}.$$

nd

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

Odd half-multiple sine series) Let $f(t)$ be given for $< t < L$, and define $F(t)$ for $0 < t < 2L$ as follows:

$$F(t) = \begin{cases} f(t), & 0 < t < L; \\ f(2L - t), & L < t < 2L. \end{cases}$$

'hus the graph of $F(t)$ is symmetric around the line $t = L$ Fig. 9.3.7). Then the period $4L$ Fourier sine series of F s

$$F(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2L},$$

vhere

$$b_n = \frac{1}{L} \int_0^L f(t) \sin \frac{n\pi t}{2L}\, dt + \frac{1}{L} \int_L^{2L} f(2L - t) \sin \frac{n\pi t}{2L}\, dt.$$

ubstitute $s = 2L - t$ in the second integral to derive the eries (for $0 < t < L$)

$$f(t) = \sum_{n \text{ odd}} b_n \sin \frac{n\pi t}{2L},$$

vhere

$$b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{2L}\, dt \quad (n \text{ odd}).$$

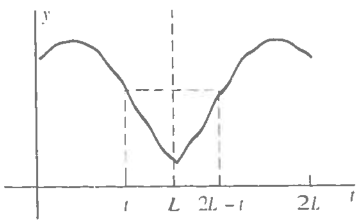


FIGURE 9.3.7. Construction of F from f in Problem 21.

Odd half-multiple cosine series) Let $f(t)$ be given for) $< t < L$, and define $G(t)$ for $0 < t < 2L$ as follows:

$$G(t) = \begin{cases} f(t), & 0 < t < L; \\ -f(2L - t), & L < t < 2L. \end{cases}$$

Jse the period $4L$ Fourier cosine series of $G(t)$ to derive he series (for $0 < t < L$)

$$f(t) = \sum_{n \text{ odd}} a_n \cos \frac{n\pi t}{2L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(t) \cos \frac{n\pi t}{2L}\, dt \qquad (n \text{ odd}).$$

23. Given: $f(t) = t$, $0 < t < \pi$. Derive the odd half-multiple sine series (Problem 21)

$$f(t) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^2} \sin \frac{nt}{2}.$$

24. Given the endpoint value problem

$$x'' - x = t, \quad x(0) = 0, \quad x'(\pi) = 0,$$

note that any constant multiple of $\sin(nt/2)$ with n odd satisfies the endpoint conditions. Hence use the odd half-multiple sine series of Problem 23 to derive the formal Fourier series solution

$$x(t) = \frac{32}{\pi} \sum_{n \text{ odd}} \frac{(-1)^{(n+1)/2}}{n^2(n^2+4)} \sin \frac{nt}{2}.$$

25. In this problem we outline the proof of Theorem 2. Suppose that $f(t)$ is a piecewise continuous period $2L$ function. Define

$$F(t) = \int_0^t \left[f(s) - \tfrac{1}{2}a_0\right] ds,$$

where $\{a_n\}$ and $\{b_n\}$ denote the Fourier coefficients of $f(t)$. **(a)** Show directly that $F(t + 2L) = F(t)$, so that F is a continuous period $2L$ function and therefore has a convergent Fourier series

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right).$$

(b) Suppose that $n \geqq 1$. Show by direct computation that

$$A_n = -\frac{L}{n\pi} b_n \quad \text{and} \quad B_n = \frac{L}{n\pi} a_n.$$

(c) Thus

$$\int_0^t f(s)\, ds = \frac{t}{2} a_0 + \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(a_n \sin \frac{n\pi t}{L} - b_n \cos \frac{n\pi t}{L} \right).$$

Finally, substitute $t = 0$ to see that

$$\frac{1}{2} A_0 = \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n.$$

9.3 Application Fourier Series of Piecewise Smooth Functions

Most computer algebra systems permit the use of unit step functions for the efficient derivation of Fourier series of "piecewise-defined" functions. Here we illustrate the use of *Maple* for this purpose. *Mathematica* and MATLAB versions can be found in the applications manual that accompanies this text.

Let the "unit function" unit(t, a, b) have the value 1 on the interval $a \leqq t < b$ and the value 0 otherwise. Then we can define a given piecewise smooth function $f(t)$ as a "linear combination" of different unit functions corresponding to the separate intervals on which the function is smooth, with the unit function for each interval multiplied by the formula defining $f(t)$ on that interval. For example, consider the even period 2π function whose graph is shown in Fig. 9.3.8. This "trapezoidal wave function" is defined for $0 < t < \pi$ by

$$f(t) = \frac{\pi}{3}\operatorname{unit}\left(t, 0, \frac{\pi}{6}\right) + \left(\frac{\pi}{2} - t\right)\operatorname{unit}\left(t, \frac{\pi}{6}, \frac{5\pi}{6}\right) + \left(-\frac{\pi}{3}\right)\operatorname{unit}\left(t, \frac{5\pi}{6}, \pi\right). \quad (1)$$

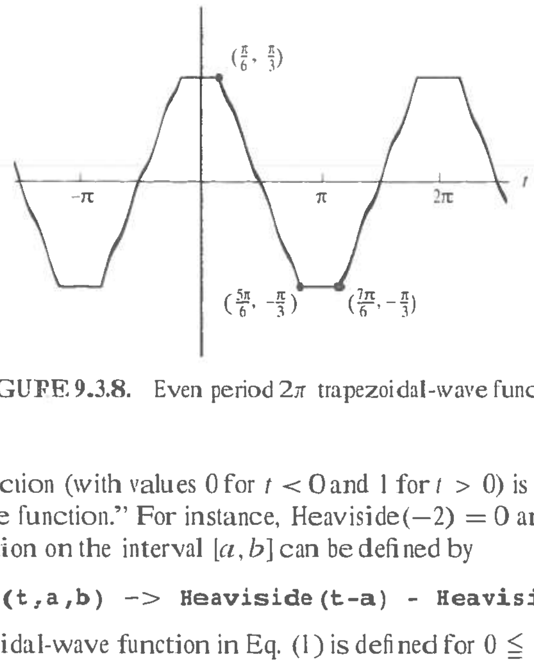


FIGURE 9.3.8. Even period 2π trapezoidal-wave function.

The unit step function (with values 0 for $t < 0$ and 1 for $t > 0$) is available in *Maple* as the "Heaviside function." For instance, Heaviside$(-2) = 0$ and Heaviside$(3) = 1$. The unit function on the interval $[a, b]$ can be defined by

```
unit := (t,a,b) -> Heaviside(t-a) - Heaviside(t-b):
```

Then the trapezoidal-wave function in Eq. (1) is defined for $0 \leqq t \leqq \pi$ by

```
f := t -> (Pi/3)*unit(t, 0, Pi/6) +
                (Pi/2 - t)*unit(t, Pi/6, 5*Pi/6) +
                (-Pi/3)*unit(t, 5*Pi/6, Pi):
```

We can now calculate the Fourier coefficients in the cosine series $f(t) = \frac{1}{2}a_0 + \sum a_n \cos nt$.

```
a := n -> (2/Pi)*int(f(t)*cos(n*t), t=0..Pi);
```

We then find that a typical partial sum of the series is given by

```
fourierSum := a(0)/2 + sum(a(n)*cos(n*t), n=1..25);
```

$$fourierSum := 2\frac{\sqrt{3}\cos(t)}{\pi} - \frac{2}{25}\frac{\sqrt{3}\cos(5t)}{\pi} - \frac{2}{49}\frac{\sqrt{3}\cos(7t)}{\pi}$$
$$+ \frac{2}{121}\frac{\sqrt{3}\cos(11t)}{\pi} + \frac{2}{169}\frac{\sqrt{3}\cos(13t)}{\pi} - \frac{2}{289}\frac{\sqrt{3}\cos(17t)}{\pi}$$
$$- \frac{2}{361}\frac{\sqrt{3}\cos(19t)}{\pi} + \frac{2}{529}\frac{\sqrt{3}\cos(23t)}{\pi} + \frac{2}{625}\frac{\sqrt{3}\cos(25t)}{\pi}$$

Thus we discover the lovely Fourier series

$$f(t) = \frac{2\sqrt{3}}{\pi}\sum \frac{(\pm)\cos nt}{n^2} \tag{2}$$

with a $+--++--++$ pattern of signs, and where the summation is taken over all odd positive integers n that are *not* multiples of 3. You can enter the command

```
plot( fourierSum, t=-2*Pi..3*Pi);
```

to verify that this Fourier series is consistent with Fig. 9.3.8.

You can then apply this method to find the Fourier series of the following period 2π functions.

1. The even square-wave function whose graph is shown in Fig. 9.3.9.
2. The even and odd triangular-wave functions whose graphs are shown in Figs. 9.2.4 and 9.3.10.
3. The odd trapezoidal-wave function whose graph is shown in Fig. 9.2.5.

Then find similarly the Fourier series of some piecewise smooth functions of your own choice, perhaps ones that have periods other than 2π and are neither even nor odd.

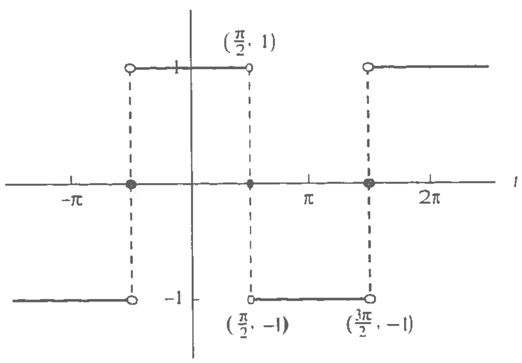


FIGURE 9.3.9. Even period 2π square-wave function.

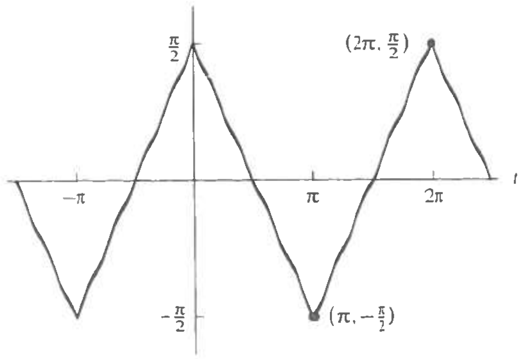


FIGURE 9.3.10. Even period 2π triangular-wave function.

4 Applications of Fourier Series

We consider first the undamped motion of a mass m on a spring with Hooke's constant k under the influence of a *periodic* external force $F(t)$, as indicated in Fig. 9.4.1. Its displacement $x(t)$ from equilibrium satisfies the familiar equation

$$mx'' + kx = F(t). \tag{1}$$

Example 2 Suppose that $m = 2$ kg and $k = 32$ N/m as in Example 1. Determine whether pure resonance will occur if $F(t)$ is the odd periodic function defined in one period to be

(a) $F(t) = \begin{cases} +10, & 0 < t < \pi; \\ -10, & \pi < t < 2\pi. \end{cases}$

(b) $F(t) = 10t, \quad -\pi < t < \pi.$

Solution (a) The natural frequency is $\omega_0 = 4$, and the Fourier series of $F(t)$ is

$$F(t) = \frac{40}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

Because this series contains no $\sin 4t$ term, no resonance occurs.

(b) In this case the Fourier series is

$$F(t) = 20\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt.$$

Pure resonance occurs because of the presence of the term containing the factor $\sin 4t$. ■

Example 3 illustrates the *near resonance* that can occur when a single term in the solution is magnified because its frequency is close to the natural frequency ω_0.

Example 3 Find a steady periodic solution of

$$x'' + 10x = F(t), \tag{10}$$

where $F(t)$ is the period 4 function with $F(t) = 5t$ for $-2 < t < 2$ and Fourier series

$$F(t) = \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}. \tag{11}$$

Solution When we substitute Eq. (11) and

$$x_{\text{sp}}(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2}$$

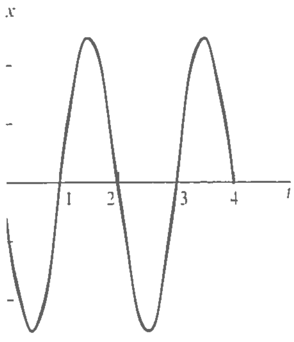
in (10), we obtain

$$\sum_{n=1}^{\infty} b_n\left(-\frac{n^2\pi^2}{4} + 10\right)\sin\frac{n\pi t}{2} = \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}.$$

We equate coefficients of like terms and then solve for b_n to get the steady periodic solution

$$x_{\text{sp}}(t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(40 - n^2\pi^2)}\sin\frac{n\pi t}{2}$$

$$\approx (0.8452)\sin\frac{\pi t}{2} - (24.4111)\sin\frac{2\pi t}{2} - (0.1738)\sin\frac{3\pi t}{2} + \cdots.$$



RE 9.4.5. The graph of

The very large magnitude of the second term results from the fact that $\omega_0 = \sqrt{10} \approx \pi = 2\pi/2$. Thus the dominant motion of a spring with the differential equation in (10) would be an oscillation with frequency π radians per second, period 2 s, and amplitude about 24, consistent with the graph of $x_{sp}(t)$ shown in Fig. 9.4.5. ■

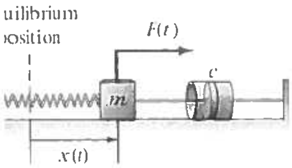
Damped Forced Oscillations

Now we consider the motion of a mass m attached both to a spring with Hooke's constant k and to a dashpot with damping constant c, under the influence of a *periodic* external force $F(t)$ (Fig. 9.4.6). The displacement $x(t)$ of the mass from equilibrium satisfies the equation

$$mx'' + cx' + kx = F(t). \tag{12}$$

We recall from Problem 25 of Section 3.6 that the steady periodic solution of Eq. (12) with $F(t) = F_0 \sin \omega t$ is

$$x(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \alpha), \tag{13}$$



RE 9.4.6. A damped and–spring system with al force.

where

$$\alpha = \tan^{-1} \frac{c\omega}{k - m\omega^2}, \quad 0 \leqq \alpha \leqq \pi. \tag{14}$$

If $F(t)$ is an odd period $2L$ function with Fourier series

$$F(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi t}{L}, \tag{15}$$

then the preceding formulas yield, by superposition, the steady periodic solution

$$x_{sp}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}}, \tag{16}$$

where $\omega_n = n\pi/L$ and α_n is the angle determined by Eq. (14) with this value of ω. Example 4 illustrates the interesting fact that the dominant frequency of the steady periodic solution can be an *integral multiple* of the frequency of the force $F(t)$.

Example 4 Suppose that $m = 3$ kg, $c = 0.02$ N/m/s, $k = 27$ N/m, and $F(t)$ is the odd period 2π function with $F(t) = \pi t - t^2$ if $0 < t < \pi$. Find the steady periodic motion $x_{sp}(t)$.

Solution We find that the Fourier series of $F(t)$ is

$$F(t) = \frac{8}{\pi}\left(\sin t + \frac{1}{3^3}\sin 3t + \frac{1}{5^3}\sin 5t + \cdots\right). \tag{17}$$

Thus $B_n = 0$ for n even, $B_n = 8/(\pi n^3)$ for n odd, and $\omega_n = n$. Equation (16) gives

$$x_{sp}(t) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt - \alpha_n)}{n^3\sqrt{(27 - 3n^2)^2 + (0.02n)^2}} \tag{18}$$

with

$$\alpha_n = \tan^{-1}\frac{(0.02)n}{27-3n^2}, \quad 0 \leqq \alpha_n \leqq \pi. \tag{19}$$

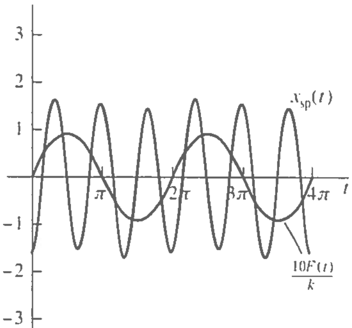


FIGURE 9.4.7. The imposed force and the resulting steady periodic motion in Example 4.

With the aid of a programmable calculator, we find that

$$\begin{aligned} x_{sp}(t) \approx{} & (0.1061)\sin(t - 0.0008) + (1.5719)\sin\left(3t - \tfrac{1}{2}\pi\right) \\ & + (0.0004)\sin(5t - 3.1437) + (0.0001)\sin(7t - 3.1428) + \cdots. \end{aligned} \tag{20}$$

Because the coefficient corresponding to $n = 3$ is much larger than the others, the response of this system is approximately a sinusoidal motion with frequency *three times* that of the input force. Figure 9.4.7 shows $x_{sp}(t)$ in comparison with the scaled force $10F(t)/k$ that has the appropriate dimension of distance.

What is happening here is this: The mass $m = 3$ on a spring with $k = 27$ has (if we ignore the small effect of the dashpot) a natural frequency $\omega_0 = \sqrt{k/m} = 3$ rad/s. The imposed external force $F(t)$ has a (smallest) period of 2π s and hence a fundamental frequency of 1 rad/s. Consequently, the term corresponding to $n = 3$ in the Fourier series of $F(t)$ (in Eq. (17)) has the same frequency as the natural frequency of the system. Thus near resonance vibrations occur, with the mass completing essentially *three* oscillations for every single oscillation of the external force. This is the physical effect of the dominant $n = 3$ term on the right-hand side in Eq. (20). For instance, you can push a friend in a swing quite high even if you push the swing only every *third* time it returns to you. This also explains why some transformers "hum" at a frequency much higher than 60 Hz.

This is a general phenomenon that must be taken into account in the design of mechanical systems. To avoid the occurrence of abnormally large and potentially destructive near resonance vibrations, the system must be so designed that it is not subject to any external periodic force, some *integral multiple* of whose fundamental frequency is close to a natural frequency of vibration. ■

Example 1 *Continued*

Finally, let us add to the mass–spring system of Example 1 a dashpot with damping constant $c = 3$ N/m/s. Then, since $m = 2$ and $k = 32$, the differential equation satisfied by the mass's displacement function $x(t)$ is now

$$2x'' + 3x' + 32x = F(t), \tag{21}$$

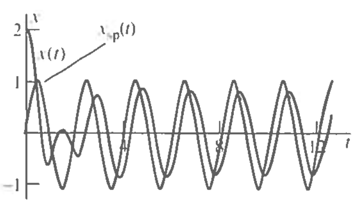


FIGURE 9.4.8. The steady periodic solution $x_{sp}(t)$ and the damped solution $x(t)$.

where $F(t)$ is the periodic force function defined in Eq. (5). Figure 9.4.8 shows graphs of both the steady periodic solution $x_{sp}(t)$ for the original undamped system of Example 1 and a numerically calculated solution of Eq. (21) with initial conditions $x(0) = 2$ and $x'(0) = 1$. As an initial transient solution determined by the initial conditions dies out, it appears that the damped solution $x(t)$ converges to a steady periodic solution of (21). However, we observe two evident effects of the damping—the amplitude of the steady periodic oscillation is decreased, and the damped steady oscillations lag behind the undamped steady oscillations. ■

9.4 Problems

ind the steady periodic solution $x_{sp}(t)$ of each of the differen-al equations in 1 through 6. Use a computer algebra system plot enough terms of the series to determine the visual ap-earance of the graph of $x_{sp}(t)$.

1. $x'' + 5x = F(t)$, where $F(t)$ is the function of period 2π such that $F(t) = 3$ if $0 < t < \pi$, $F(t) = -3$ if $\pi < t < 2\pi$.
2. $x'' + 10x = F(t)$, where $F(t)$ is the even function of period 4 such that $F(t) = 3$ if $0 < t < 1$, $F(t) = -3$ if $1 < t < 2$.
3. $x'' + 3x = F(t)$, where $F(t)$ is the odd function of period 2π such that $F(t) = 2t$ if $0 < t < \pi$.
4. $x'' + 4x = F(t)$, where $F(t)$ is the even function of period 4 such that $F(t) = 2t$ if $0 < t < 2$.
5. $x'' + 10x = F(t)$, where $F(t)$ is the odd function of period 2 such that $F(t) = t - t^2$ if $0 < t < 1$.
6. $x'' + 2x = F(t)$, where $F(t)$ is the even function of period 2π such that $F(t) = \sin t$ if $0 < t < \pi$.

each of Problems 7 through 12, the mass m and Hooke's onstant k for a mass-and-spring system are given. Determine hether or not pure resonance will occur under the influence f the given external periodic force $F(t)$.

7. $m = 1$, $k = 9$; $F(t)$ is the odd function of period 2π with $F(t) = 1$ for $0 < t < \pi$.
8. $m = 2$, $k = 10$; $F(t)$ is the odd function of period 2 with $F(t) = 1$ for $0 < t < 1$.
9. $m = 3$, $k = 12$; $F(t)$ is the odd function of period 2π with $F(t) = 3$ for $0 < t < \pi$.
10. $m = 1$, $k = 4\pi^2$; $F(t)$ is the odd function of period 2 with $F(t) = 2t$ for $0 < t < 1$.
11. $m = 3$, $k = 48$; $F(t)$ is the even function of period 2π with $F(t) = t$ for $0 < t < \pi$.
12. $m = 2$, $k = 50$; $F(t)$ is the odd function of period 2π with $F(t) = \pi t - t^2$ for $0 < t < \pi$.

In each of Problems 13 through 16, the values of m, c, and k for a damped mass-and-spring system are given. Find the steady periodic motion—in the form of Eq. (16)—of the mass under the influence of the given external force $F(t)$. Compute the coefficients and phase angles for the first three nonzero terms in the series for $x_{sp}(t)$.

13. $m = 1$, $c = 0.1$, $k = 4$; $F(t)$ is the force of Problem 1.
14. $m = 2$, $c = 0.1$, $k = 18$; $F(t)$ is the force of Problem 3.
15. $m = 3$, $c = 1$, $k = 30$; $F(t)$ is the force of Problem 5.
16. $m = 1$, $c = 0.01$, $k = 4$; $F(t)$ is the force of Problem 4.
17. Consider a forced damped mass-and-spring system with $m = \frac{1}{4}$ slug, $c = 0.6$ lb/ft/s, $k = 36$ lb/ft. The force $F(t)$ is the period 2 (s) function with $F(t) = 15$ if $0 < t < 1$, $F(t) = -15$ if $1 < t < 2$. **(a)** Find the steady periodic solution in the form
$$x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t - \alpha_n).$$
(b) Find the location—to the nearest tenth of an inch—of the mass when $t = 5$ s.
18. Consider a forced damped mass-and-spring system with $m = 1$, $c = 0.01$, and $k = 25$. The force $F(t)$ is the odd function of period 2π with $F(t) = t$ if $0 < t < \pi/2$, $F(t) = \pi - t$ if $\pi/2 < t < \pi$. Find the steady periodic motion; compute enough terms of its series to see that the dominant frequency of the motion is *five* times that of the external force.
19. Suppose the functions $f(t)$ and $g(t)$ are periodic with periods P and Q, respectively. If the ratio P/Q of their periods is a rational number, show that the sum $f(t) + g(t)$ is a periodic function.
20. If p/q is irrational, prove that the function $f(t) = \cos pt + \cos qt$ is not a periodic function. *Suggestion*: Show that the assumption $f(t+L) = f(t)$ would (upon substituting $t = 0$) imply that p/q is rational.

9.5 Heat Conduction and Separation of Variables

The most important applications of Fourier series are to the solution of partial differential equations by means of the method of separation of variables that we introduce in this section. Recall that a *partial differential equation* is one containing one or more *partial* derivatives of a dependent variable that is a function of at least two independent variables. An example is the **one-dimensional heat equation**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

in which the dependent variable u is an unknown function of x and t, and k is a given positive constant.

The Heated Rod

Equation (1) models the variation of temperature u with position x and time t in a heated rod that extends along the x-axis. We assume that the rod has uniform cross section with area A perpendicular to the axis and that it is made of a homogeneous material. We assume further that the cross section of the rod is so small that u is constant on each cross section, and that the lateral surface of the rod is insulated so that no heat can pass through it. Then u will, indeed, be a function of x and t, and heat will flow along the rod in only the x-direction. In general, we envision heat as flowing like a fluid from the warmer to the cooler parts of a body.

The **heat flux** $\phi(x, t)$ in the rod is the *rate of flow of heat* (in the positive x-direction) at time t across a unit area of the rod's cross section at x. Typical units for ϕ are calories (of heat) per second per square centimeter (of area). The derivation of Eq. (1) is based on the empirical principle that

$$\phi = -K\frac{\partial u}{\partial x}, \tag{2}$$

where the positive proportionality constant K is called the **thermal conductivity** of the material of the rod. Note that if $u_x > 0$, then $\phi < 0$, meaning that heat is flowing in the negative x-direction, while if $u_x < 0$, then $\phi > 0$, so heat is flowing in the positive x-direction. Thus the *rate* of heat flow is proportional to $|u_x|$, and the *direction* of heat flow is in the direction along the rod in which the temperature u is decreasing. In short, heat flows from a warm place to a cool place, not vice versa.

Now consider a small segment of the rod corresponding to the interval $[x, x + \Delta x]$, as shown in Fig. 9.5.1. The rate of flow R (in calories per second) of heat *into* this segment through its two ends is

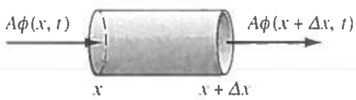


FIGURE 9.5.1. Net flow of heat into a short segment of the rod.

$$R = A\phi(x, t) - A\phi(x + \Delta x, t) = KA[u_x(x + \Delta x, t) - u_x(x, t)]. \tag{3}$$

The resulting time rate of change u_t of the temperature in the segment depends on its density δ (grams per cubic centimeter) and specific heat c (both assumed constant). The **specific heat** c is the amount of heat (in calories) required to raise by 1° (Celsius) the temperature of 1 g of material. Consequently $c\delta u$ calories of heat are required to raise 1 cm^3 of the material from temperature zero to temperature u. A short slice of the rod of length dx has volume $A\,dx$, so $c\delta u A\,dx$ calories of heat are required to raise the temperature of this slice from 0 to u. The **heat content**

$$Q(t) = \int_x^{x+\Delta x} c\delta A u(x, t)\,dx \tag{4}$$

of the segment $[x, x + \Delta x]$ of the rod is the amount of heat needed to raise it from zero temperature to the given temperature $u(x, t)$. Because heat enters and leaves the segment only through its ends, we see from Eq. (3) that

$$Q'(t) = KA[u_x(x + \Delta x, t) - u_x(x, t)], \tag{5}$$

because $R = dQ/dt$. Thus by differentiating Eq. (4) within the integral and applying the mean value theorem for integrals, we see that

$$Q'(t) = \int_x^{x+\Delta x} c\delta A u_t(x, t)\,dx = c\delta A u_t(\overline{x}, t)\,\Delta x \tag{6}$$

for some $\overline{x}$ in $(x, x + \Delta x)$. Upon equating the values in Eqs. (5) and (6), we get

$$c\delta A u_t(\overline{x}, t)\,\Delta x = KA[u_x(x + \Delta x, t) - u_x(x, t)], \tag{7}$$

so

$$u_t(\overline{x}, t) = k\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x}, \tag{8}$$

where

$$k = \frac{K}{c\delta} \tag{9}$$

is the **thermal diffusivity** of the material. We now take the limit as $\Delta x \to 0$, so $\overline{x} \to x$ (because $\overline{x}$ lies in the interval $[x, x + \Delta x]$ with fixed left endpoint x). Then the two sides of the equation in (8) approach the two sides of the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Thus the temperature $u(x, t)$ in the thin rod with insulated sides must satisfy this partial differential equation.

Boundary Conditions

Now suppose that the rod has finite length L, extending from $x = 0$ to $x = L$. Its temperature function $u(x, t)$ will be determined among all possible solutions of Eq. (1) by appropriate subsidiary conditions. In fact, whereas a solution of an ordinary differential equation involves arbitrary *constants*, a solution of a partial differential equation generally involves arbitrary *functions*. In the case of the heated rod, we can specify its temperature function $f(x)$ at time $t = 0$. This gives the *initial condition*

$$u(x, 0) = f(x). \tag{10}$$

We may also specify fixed temperatures at the two ends of the rod. For instance, if each end were clamped against a large block of ice at temperature zero, we would have the *endpoint conditions*

$$u(0, t) = u(L, t) = 0 \qquad (\text{for all } t > 0). \tag{11}$$

Combining all this, we get the **boundary value problem**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\quad t > 0); \tag{12a}$$

$$u(0, t) = u(L, t) = 0, \qquad (t > 0), \tag{12b}$$

$$u(x, 0) = f(x) \qquad (0 < x < L). \tag{12c}$$

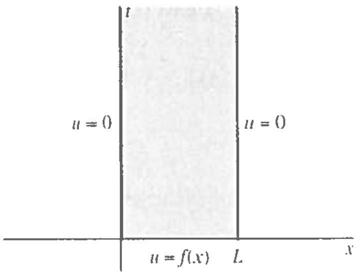


FIGURE 9.5.2. A geometric interpretation of the boundary value problem in Eqs. (12a)–(12c).

Figure 9.5.2 gives a geometric interpretation of the boundary value problem in (12): We are to find a function $u(x,t)$ that is continuous on the unbounded strip (including its boundary) shaded in the xt-plane. This function must satisfy the differential equation in (12a) at each interior point of the strip, and on the boundary of the strip must have the values prescribed by the *boundary conditions* in (12b) and (12c). Physical intuition suggests that if $f(x)$ is a reasonable function, then there will exist one and only one such function $u(x,t)$.

Instead of having fixed temperatures, the two ends of the rod might be insulated. In this case no heat would flow through either end, so we see from Eq. (2) that the conditions in (12b) would be replaced in the boundary value problem by the endpoint conditions

$$u_x(0,t) = u_x(L,t) = 0 \tag{13}$$

(for all t). Alternatively, the rod could be insulated at one end and have a fixed temperature at the other. This and other endpoint possibilities are discussed in the Problems.

Superposition of Solutions

Note that the heat equation in (12a) is *linear*. That is, any linear combination $u = c_1u_1 + c_2u_2$ of two solutions of (12a) is also a solution of (12a); this follows immediately from the linearity of partial differentiation. It is also true that if u_1 and u_2 each satisfy the conditions in (12b), then so does any linear combination $u = c_1u_1 + c_2u_2$. The conditions in (12b) are therefore called **homogeneous** boundary conditions (though a more descriptive term might be *linear* boundary conditions). By contrast, the final boundary condition in (12c) is not homogeneous; it is a **nonhomogeneous** boundary condition.

Our overall strategy for solving the boundary value problem in (12) will be to find functions $u_1, u_2, u_3, \ldots$ that satisfy both the partial differential equation in (12a) and the homogeneous boundary conditions in (12b), and then attempt to combine these functions by superposition, as if they were building blocks, in the hope of obtaining a solution $u = c_1u_1 + c_2u_2 + \cdots$ that satisfies the nonhomogeneous condition in (12c) as well. Example 1 illustrates this approach.

Example 1 It is easy to verify by direct substitution that each of the functions

$$u_1(x,t) = e^{-t}\sin x, \quad u_2(x,t) = e^{-4t}\sin 2x, \quad \text{and} \quad u_3(x,t) = e^{-9t}\sin 3x$$

satisfies the equation $u_t = u_{xx}$. Use these functions to construct a solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad t > 0); \tag{14a}$$

$$u(0,t) = u(\pi,t) = 0, \tag{14b}$$

$$u(x,0) = 80\sin^3 x = 60\sin x - 20\sin 3x. \tag{14c}$$

Solution Any linear combination of the form

$$\begin{aligned} u(x,t) &= c_1u_1(x,t) + c_2u_2(x,t) + c_3u_3(x,t) \\ &= c_1e^{-t}\sin x + c_2e^{-4t}\sin 2x + c_3e^{-9t}\sin 3x \end{aligned}$$

satisfies both the differential equation in (14a) and the homogeneous conditions in (14b). Because

$$u(x, 0) = c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x,$$

we see that we can also satisfy the nonhomogeneous condition in (14c) simply by choosing $c_1 = 60$, $c_2 = 0$, and $c_3 = -20$. Thus a solution of the given boundary value problem is

$$u(x, t) = 60e^{-t} \sin x - 20e^{-9t} \sin 3x.$$ ■

The boundary value problem in Example 1 is exceptionally simple in that only a finite number of homogeneous solutions are needed to satisfy by superposition the nonhomogeneous boundary condition. It is more usual that an infinite sequence u_1, u_2, u_3, ... of functions satisfying (12a) and (12b) is required. If so, we write the infinite series

➤ $$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) \tag{15}$$

and then attempt to determine the coefficients $c_1, c_2, c_3, \ldots$ in order to satisfy (12c) as well. The following principle summarizes the properties of this infinite series that must be verified to ensure that we have a solution of the boundary value problem in (12).

PRINCIPLE Superposition of Solutions

Suppose that each of the functions u_1, u_2, u_3, ... satisfies both the differential equation in (12a) (for $0 < x < L$ and $t > 0$) and the homogeneous conditions in (12b). Suppose also that the coefficients in Eq. (15) are chosen to meet the following three criteria:

1. For $0 < x < L$ and $t > 0$, the function determined by the series in (15) is continuous and termwise differentiable (once with respect to t and twice with respect to x).
2. $\sum_{n=1}^{\infty} c_n u_n(x, 0) = f(x)$ for $0 < x < L$.
3. The function $u(x, t)$ determined by Eq. (15) interior to the strip $0 \leqq x \leqq L$ and $t \geqq 0$, and by the boundary conditions in (12b) and (12c) on its boundary, is continuous.

Then $u(x, t)$ is a solution of the boundary value problem in (12).

In the method of separation of variables described next, we concentrate on finding the solutions u_1, u_2, u_3, ... satisfying the homogeneous conditions and on determining the coefficients so that the series in Eq. (15) satisfies the nonhomogeneous conditions upon substitution of $t = 0$. At this point we have only a *formal* series solution of the boundary value problem—one that is subject to verification of the continuity and differentiability conditions given in part (1) of the superposition principle stated here. If the function $f(x)$ in (12c) is piecewise smooth, it can be proved that a formal series solution always satisfies the restrictions and, moreover, is the unique solution of the boundary value problem. For a proof, see Chapter 6 of R. V. Churchill and J. W. Brown, *Fourier Series and Boundary Value Problems*, 3rd ed. (New York: McGraw-Hill, 1978).

Separation of Variables

This method of solving the boundary value problem in (12) for the heated rod was introduced by Fourier in his study of heat cited in Section 9.1. We first search for the *building block* functions $u_1, u_2, u_3, \ldots$ that satisfy the differential equation $u_t = ku_{xx}$ and the homogeneous conditions $u(0, t) = u(L, t) = 0$, with each of these functions being of the special form

$$u(x, t) = X(x)T(t) \tag{16}$$

in which the variables are "separated"—that is, each of the building-block functions is a product of a function of position x (only) and a function of time t (only). Substitution of (16) in $u_t = ku_{xx}$ yields $XT' = kX''T$, where for brevity we write T' for $T'(t)$ and X'' for $X''(x)$. Division of both sides by kXT then gives

$$\frac{X''}{X} = \frac{T'}{kT}. \tag{17}$$

The left-hand side of Eq. (17) is a function of x alone, but the right-hand side is a function of t alone. If t is held constant on the right-hand side, then the left-hand side X''/X must remain constant as x varies. Similarly, if x is held constant on the left-hand side, then the right-hand side T'/kT must remain constant as t varies. Consequently, equality can hold only if each of these two expressions is the same *constant*, which for convenience we denote by $-\lambda$. Thus Eq. (17) becomes

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda, \tag{18}$$

which consists of the two equations

$$X''(x) + \lambda X(x) = 0, \tag{19}$$

$$T'(t) + \lambda kT(t) = 0. \tag{20}$$

It follows that the product function $u(x, t) = X(x)T(t)$ satisfies the partial differential equation $u_t = ku_{xx}$ if $X(x)$ and $T(t)$ separately satisfy the ordinary differential equations in (19) and (20) for some (common) value of the constant λ.

We focus first on $X(x)$. The homogeneous endpoint conditions are

$$u(x, 0) = X(0)T(t) = 0, \qquad u(L, t) = X(L)T(t) = 0. \tag{21}$$

If $T(t)$ is to be a nontrivial function of t, then (21) can hold only if $X(0) = X(L) = 0$. Thus $X(x)$ must satisfy the endpoint value problem

$$\begin{gathered} X'' + \lambda X = 0, \\ X(0) = 0, \quad X(L) = 0. \end{gathered} \tag{22}$$

This is actually an eigenvalue problem of the type we discussed in Section 3.8. Indeed, we saw in Example 3 of that section that (22) has a nontrivial solution if and only if λ is one of the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, 3, \ldots, \tag{23}$$

and that an eigenfunction associated with λ_n is

$$X_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots. \tag{24}$$

Recall that the reasoning behind (23) and (24) is as follows. If $\lambda = 0$, then (22) obviously implies that $X(x) \equiv 0$. If $\lambda = -\alpha^2 < 0$, then

$$X(x) = A\cosh\alpha x + B\sinh\alpha x,$$

and then the conditions $X(0) = 0 = X(L)$ imply that $A = B = 0$. Hence the only possibility for a nontrivial eigenfunction is that $\lambda = \alpha^2 > 0$. Then

$$X(x) = A\cos\alpha x + B\sin\alpha x,$$

and the conditions $X(0) = 0 = X(L)$ then imply that $A = 0$ and that $\alpha = n\pi/L$ for some positive integer n. (Whenever separation of variables leads to an unfamiliar eigenvalue problem, we generally must consider separately the cases $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$.)

Now we turn our attention to Eq. (20), knowing that the constant λ must be one of the eigenvalues listed in (23). For the nth of these possibilities we write Eq. (20) as

$$T_n' + \frac{n^2\pi^2 k}{L^2}T_n = 0, \tag{25}$$

in anticipation of a different solution $T_n(t)$ for each different positive integer n. A nontrivial solution of this equation is

$$T_n(t) = \exp\left(-n^2\pi^2 kt/L^2\right). \tag{26}$$

We omit the arbitrary constant of integration because it will (in effect) be inserted later.

To summarize our progress, we have discovered the two associated sequences $\{X_n\}_1^\infty$ and $\{T_n\}_1^\infty$ of functions given in (24) and (26). Together they yield the sequence of building-block product functions

$$u_n(x, t) = X_n(x)T_n(t) = \exp\left(-n^2\pi^2 kt/L^2\right)\sin\frac{n\pi x}{L}, \tag{27}$$

$n = 1, 2, 3, \ldots$. Each of these functions satisfies both the heat equation $u_t = ku_{xx}$ and the homogeneous conditions $u(0, t) = u(L, t) = 0$. Now we combine these functions (*superposition*) to attempt to satisfy the nonhomogeneous condition $u(x, 0) = f(x)$ as well. We therefore form the infinite series

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{n=1}^{\infty} c_n \exp\left(-n^2\pi^2 kt/L^2\right)\sin\frac{n\pi x}{L}. \tag{28}$$

It remains only to determine the constant coefficients $\{c_n\}_1^\infty$ so that

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x) \tag{29}$$

for $0 < x < L$. But this will be the Fourier series of $f(x)$ on $[0, L]$ provided that we choose

$$➤ \qquad c_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{30}$$

for each $n = 1, 2, 3, \ldots$. Thus we have the following result.

THEOREM 1 The Heated Rod with Zero Endpoint Temperatures

The boundary value problem in (12) for a heated rod with zero endpoint temperatures has the formal series solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-n^2\pi^2kt/L^2\right) \sin\frac{n\pi x}{L}, \tag{31}$$

where the $\{b_n\}$ are the Fourier sine coefficients in Eq. (30) of the rod's initial temperature function $f(x) = u(x, 0)$.

Material	k (cm²/s)
Silver	1.70
Copper	1.15
Aluminum	0.85
Iron	0.15
Concrete	0.005

FIGURE 9.5.3. Some thermal diffusivity constants.

Remark: By taking the limit in (31) termwise as $t \to \infty$, we get $u(x, \infty) \equiv 0$, as we expect because the two ends of the rod are held at temperature zero. ■

The series solution in Eq. (31) usually converges quite rapidly, unless t is very small, because of the presence of the negative exponential factors. Therefore it is practical for numerical computations. For use in problems and examples, values of the thermal diffusivity constant k for some common materials are listed in the table in Fig. 9.5.3.

Example 2 Suppose that a rod of length $L = 50$ cm is immersed in steam until its temperature is $u_0 = 100°$C throughout. At time $t = 0$, its lateral surface is insulated and its two ends are imbedded in ice at 0°C. Calculate the rod's temperature at its midpoint after half an hour if it is made of (a) iron; (b) concrete.

Solution The boundary value problem for this rod's temperature function $u(x, t)$ is

$$\begin{aligned} u_t &= ku_{xx}, \\ u(0,t) &= u(L,t) = 0; \\ u(x,0) &= u_0. \end{aligned}$$

Recall the square-wave series

$$f(t) = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{1}{n}\sin\frac{n\pi t}{L} = \begin{cases} +1 & \text{if } 0 < t < L, \\ -1 & \text{if } -L < t < 0 \end{cases}$$

that we derived in Example 1 of Section 9.2. It follows that the Fourier sine series of $f(x) \equiv u_0$ is

$$f(x) = \frac{4u_0}{\pi}\sum_{n\text{ odd}} \frac{1}{n}\sin\frac{n\pi x}{L}$$

for $0 < x < L$. Hence the Fourier coefficients in Eq. (31) are given by

$$b_n = \begin{cases} \dfrac{4u_0}{n\pi} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases}$$

and therefore the rod's temperature function is given by

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n\text{ odd}} \frac{1}{n}\exp\left(-\frac{n^2\pi^2kt}{L^2}\right)\sin\frac{n\pi x}{L}.$$

Figure 9.5.4 shows a graph $u = u(x, t)$ with $u_0 = 100$ and $L = 50$. As t increases, we see the maximum temperature of the rod (evidently at its midpoint) steadily decreasing. The temperature at the midpoint $x = 25$ after $t = 1800$ seconds is

$$u(25, 1800) = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{(-1)^{n+1}}{n} \exp\left(-\frac{18n^2\pi^2 k}{25}\right).$$

(a) With the value $k = 0.15$ that was used in Fig. 9.5.4, this series gives

$$u(25, 1800) \approx 43.8519 - 0.0029 + 0.0000 - \cdots \approx 43.85°\text{C}.$$

This value $u(25, 1800) \approx 43.85$ is the maximum height (at its midpoint $x = 25$) of the vertical sectional curve $u = u(x, 1800)$ that we see at one "end" of the temperature surface shown in Fig. 9.5.4.
(b) With $k = 0.005$ for concrete, it gives

$$\begin{aligned} u(25, 1800) \approx\ & 122.8795 - 30.8257 + 10.4754 - 3.1894 \\ & + 0.7958 - 0.1572 + 0.0242 - 0.0029 \\ & + 0.0003 - 0.0000 + \cdots \approx 100.00°\text{C}. \end{aligned}$$

Thus concrete is a very effective insulator. ■

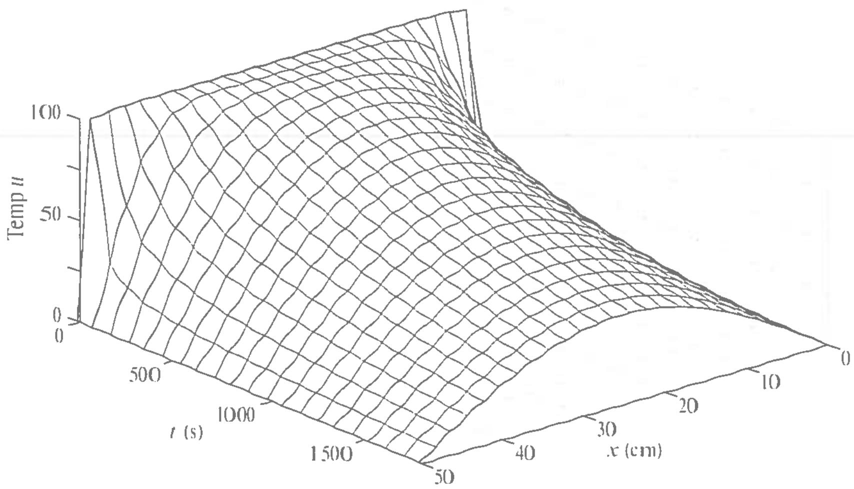


FIGURE 9.5.4. The graph of the temperature function $u(x, t)$ in Example 2.

Insulated Endpoint Conditions

We now consider the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \quad t > 0); \tag{32a}$$

$$u_x(0, t) = u_x(L, t) = 0, \tag{32b}$$

$$u(x, 0) = f(x), \tag{32c}$$

which corresponds to a rod of length L with initial temperature $f(x)$, but with its two ends insulated. The separation of variables $u(x, t) = X(x)T(t)$ proceeds as in

Eqs. (16) through (20) without change. But the homogeneous endpoint conditions in (32b) yield $X'(0) = X'(L) = 0$. Thus $X(x)$ must satisfy the endpoint value problem

$$\begin{aligned} &X'' + \lambda X = 0; \\ &X'(0) = 0, \quad X'(L) = 0. \end{aligned} \tag{33}$$

We must again consider separately the possibilities $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$ for the eigenvalues.

With $\lambda = 0$, the general solution of $X'' = 0$ is $X(x) = Ax + B$, so $X'(x) = A$. Hence the endpoint conditions in (33) require $A = 0$, but B may be nonzero. Because a constant multiple of an eigenfunction is an eigenfunction, we can choose any constant value we wish for B. Thus, with $B = 1$, we have the zero eigenvalue and associated eigenfunction

$$\lambda_0 = 0, \quad X_0(x) \equiv 1. \tag{34}$$

With $\lambda = 0$ in Eq. (20), we get $T'(t) = 0$, so we may take $T_0(t) \equiv 1$ as well.

With $\lambda = -\alpha^2 < 0$, the general solution of the equation $X'' - \alpha^2 X = 0$ is

$$X(x) = A\cosh\alpha x + B\sinh\alpha x,$$

and we readily verify that $X'(0) = X'(L) = 0$ only if $A = B = 0$. Thus there are no negative eigenvalues.

With $\lambda = \alpha^2 > 0$, the general solution of $X'' + \alpha^2 X = 0$ is

$$\begin{aligned} X(x) &= A\cos\alpha x + B\sin\alpha x, \\ X'(x) &= -A\alpha\sin\alpha x + B\alpha\cos\alpha x. \end{aligned}$$

Hence $X'(0) = 0$ implies that $B = 0$, and then

$$X'(L) = -A\alpha\sin\alpha L = 0$$

requires that αL be an integral multiple of π, because $\alpha \neq 0$ and $A \neq 0$ if we are to have a nontrivial solution. Thus we have the infinite sequence of eigenvalues and associated eigenfunctions

$$\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \cos\frac{n\pi x}{L} \tag{35}$$

for $n = 1, 2, 3, \ldots$. Just as before, the solution of Eq. (20) with $\lambda = n^2\pi^2/L^2$ is $T_n(t) = \exp\left(-n^2\pi^2kt/L^2\right)$.

Therefore, the product functions satisfying the homogeneous conditions are

$$u_0(x, t) \equiv 1; \quad u_n(x, t) = \exp\left(n^2\pi^2kt/L^2\right)\cos\frac{n\pi x}{L} \tag{36}$$

for $n = 1, 2, 3, \ldots$. Hence the trial solution is

$$u(x, t) = c_0 + \sum_{n=1}^{\infty} c_n \exp\left(-n^2\pi^2kt/L^2\right)\cos\frac{n\pi x}{L}. \tag{37}$$

To satisfy the nonhomogeneous condition $u(x, 0) = f(x)$, we obviously want Eq. (37) to reduce when $t = 0$ to the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \tag{38}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \tag{39}$$

for $n = 0, 1, 2, \ldots$. Thus we have the following result.

THEOREM 2 Heated Rod with Insulated Ends

The boundary value problem in (32) for a heated rod with insulated ends has the formal series solution

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-n^2\pi^2 kt/L^2\right) \cos \frac{n\pi x}{L} \tag{40}$$

where the $\{a_n\}$ are the Fourier cosine coefficients in (39) of the rod's initial temperature function $f(x) = u(x, 0)$.

Remark: Note that

$$\lim_{t\to\infty} u(x, t) = \frac{a_0}{2} = \frac{1}{L} \int_0^L f(x)\, dx, \tag{41}$$

the average value of the initial temperature. With both the lateral surface and the ends of the rod insulated, its original heat content ultimately distributes itself uniformly throughout the rod. ■

Example 3

We consider the same 50-cm rod as in Example 2, but now suppose that its initial temperature is given by the "triangular function" graphed in Fig. 9.5.5. At time $t = 0$, the rod's lateral surface *and* its two ends are insulated. Then its temperature function $u(x, t)$ satisfies the boundary value problem

$$u_t = ku_{xx},$$
$$u_x(0, t) = u_x(50, t) = 0,$$
$$u(x, 0) = f(x).$$

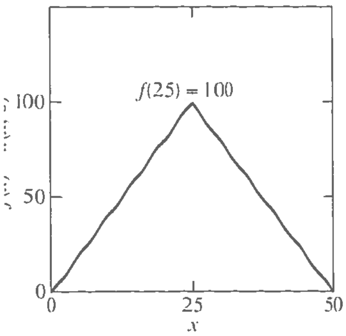


FIGURE 9.5.5. The graph of the initial temperature function $u(x, 0) = f(x)$ in Example 3.

Now substitution of $L = 25$ in the even triangular-wave series of Eq. (15) in Section 9.3 (where the length of the interval is denoted by $2L$), followed by multiplication by 4, yields the Fourier cosine series

$$f(x) = 50 - \frac{400}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{n\pi x}{25}$$

(for $0 < x < 50$) of our initial temperature function. But in order to match terms with the series in (40) with $L = 50$, we need to exhibit terms of the form

$\cos(n\pi x/50)$ rather than terms of the form $\cos(n\pi x/25)$. Hence we replace n with $n/2$ throughout and thereby rewrite the series in the form

$$f(x) = 50 - \frac{1600}{\pi^2} \sum_{n=2,6,10,\dots} \frac{1}{n^2} \cos \frac{n\pi x}{50};$$

note that the summation runs through all positive integers of the form $4m - 2$. Then Theorem 2 implies that the rod's temperature function is given by

$$u(x,t) = 50 - \frac{1600}{n^2} \sum_{n=2,6,10,\dots} \frac{1}{n^2} \exp\left(-\frac{n^2\pi^2 kt}{2500}\right) \cos \frac{n\pi x}{50}.$$

Figure 9.5.6 shows the graph $u = u(x, t)$ for the first 1200 seconds, and we see the temperature in the rod beginning with a sharp maximum at the midpoint $x = 25$, but rapidly "averaging out" as the heat in the rod is redistributed with increasing t. ■

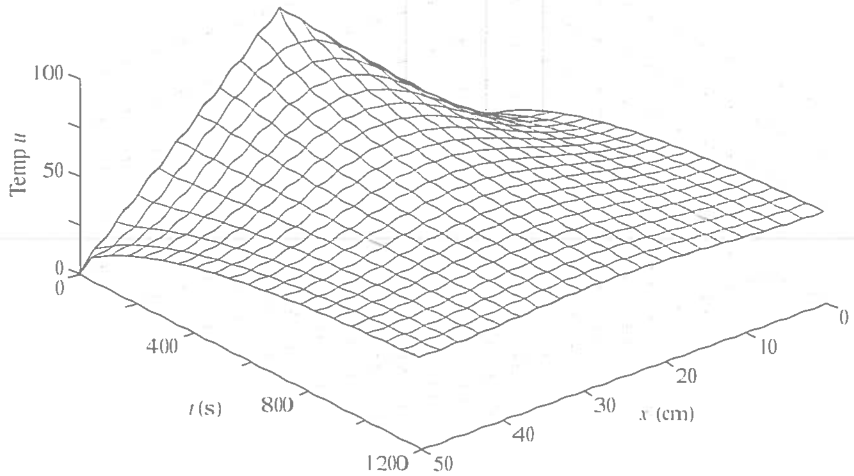


FIGURE 9.5.6. The graph of the temperature function $u(x, t)$ in Example 3.

Finally, we point out that, although we set up the boundary value problems in (12) and (32) for a rod of length L, they also model the temperature $u(x, t)$ within the infinite slab $0 \leqq x \leqq L$ in three-dimensional space if its initial temperature $f(x)$ depends only on x and its two faces $x = 0$ and $x = L$ are either both insulated or both held at temperature zero.

9.5 Problems

Solve the boundary value problems in Problems 1 through 12.

1. $u_t = 3u_{xx}$, $0 < x < \pi$, $t > 0$; $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = 4\sin 2x$

2. $u_t = 10u_{xx}$, $0 < x < 5$, $t > 0$; $u_x(0, t) = u_x(5, t) = 0$, $u(x, 0) = 7$

3. $u_t = 2u_{xx}$, $0 < x < 1$, $t > 0$; $u(0, t) = u(1, t) = 0$, $u(x, 0) = 5\sin \pi x - \frac{1}{5}\sin 3\pi x$

4. $u_t = u_{xx}$, $0 < x < \pi$, $t > 0$; $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = 4\sin 4x \cos 2x$

5. $u_t = 2u_{xx}$, $0 < x < 3$, $t > 0$; $u_x(0, t) = u_x(3, t) = 0$, $u(x, 0) = 4\cos\frac{2}{3}\pi x - 2\cos\frac{4}{3}\pi x$

6. $2u_t = u_{xx}$, $0 < x < 1$, $t > 0$; $u(0, t) = u(1, t) = 0$, $u(x, 0) = 4\sin \pi x \cos^3 \pi x$

7. $3u_t = u_{xx}$, $0 < x < 2$, $t > 0$; $u_x(0,t) = u_x(2,t) = 0$, $u(x,0) = \cos^2 2\pi x$

8. $u_t = u_{xx}$, $0 < x < 2$, $t > 0$; $u_x(0,t) = u_x(2,t) = 0$, $u(x,0) = 10\cos \pi x \cos 3\pi x$

9. $10u_t = u_{xx}$, $0 < x < 5$, $t > 0$; $u(0,t) = u(5,t) = 0$, $u(x,0) = 25$

10. $5u_t = u_{xx}$, $0 < x < 10$, $t > 0$; $u(0,t) = u(10,t) = 0$, $u(x,0) = 4x$

11. $5u_t = u_{xx}$, $0 < x < 10$, $t > 0$; $u_x(0,t) = u_x(10,t) = 0$, $u(x,0) = 4x$

12. $u_t = u_{xx}$, $0 < x < 100$, $t > 0$; $u(0,t) = u(100,t) = 0$, $u(x,0) = x(100 - x)$

13. Suppose that a rod 40 cm long with insulated lateral surface is heated to a uniform temperature of 100°C, and that at time $t = 0$ its two ends are embedded in ice at 0°C. **(a)** Find the formal series solution for the temperature $u(x,t)$ of the rod. **(b)** In the case the rod is made of copper, show that after 5 min the temperature at its midpoint is about 15°C. **(c)** In the case the rod is made of concrete, use the first term of the series to find the time required for its midpoint to cool to 15°C.

14. A copper rod 50 cm long with insulated lateral surface has initial temperature $u(x,0) = 2x$, and at time $t = 0$ its two ends are insulated. **(a)** Find $u(x,t)$. **(b)** What will its temperature be at $x = 10$ after 1 min? **(c)** After approximately how long will its temperature at $x = 10$ be 45°C?

15. The two faces of the slab $0 \leqq x \leqq L$ are kept at temperature zero, and the initial temperature of the slab is given by $u(x,0) = A$ (a constant) for $0 < x < L/2$, $u(x,0) = 0$ for $L/2 < x < L$. Derive the formal series solution

$$u(x,t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi/4)}{n}\exp\left(-n^2\pi^2kt/L^2\right)\sin\frac{n\pi x}{L}.$$

16. Two iron slabs are each 25 cm thick. Initially one is at temperature 100°C throughout and the other is at temperature 0°C. At time $t = 0$ they are placed face to face, and their outer faces are kept at 0°C. **(a)** Use the result of Problem 15 to verify that after a half hour the temperature of their common face is approximately 22°C. **(b)** Suppose that the two slabs are instead made of concrete. How long will it be until their common face reaches a temperature of 22°C?

17. (Steady-state and transient temperatures) Let a laterally insulated rod with initial temperature $u(x,0) = f(x)$ have fixed endpoint temperatures $u(0,t) = A$ and $u(L,t) = B$. **(a)** It is observed empirically that as $t \to +\infty$, $u(x,t)$ approaches a **steady-state temperature** $u_{ss}(x)$ that corresponds to setting $u_t = 0$ in the boundary value problem. Thus $u_{ss}(x)$ is the solution of the endpoint value problem

$$\frac{\partial^2 u_{ss}}{\partial x^2} = 0; \quad u_{ss}(0) = A, \quad u_{ss}(L) = B.$$

Find $u_{ss}(x)$. **(b)** The **transient temperature** $u_{tr}(x,t)$ is defined to be

$$u_{tr}(x,t) = u(x,t) - u_{ss}(x).$$

Show that u_{tr} satisfies the boundary value problem

$$\frac{\partial u_{tr}}{\partial t} = k\frac{\partial^2 u_{tr}}{\partial x^2};$$
$$u_{tr}(0,t) = u_{tr}(L,t) = 0,$$
$$u_{tr}(x,0) = g(x) = f(x) - u_{ss}(x).$$

(c) Conclude from the formulas in (30) and (31) that

$$\begin{aligned} u(x,t) &= u_{ss}(x) + u_{tr}(x,t) \\ &= u_{ss}(x) + \sum_{n=1}^{\infty} c_n \exp\left(-n^2\pi^2kt/L^2\right)\sin\frac{n\pi x}{L}, \end{aligned}$$

where

$$c_n = \frac{2}{L}\int_0^L [f(x) - u_{ss}(x)]\sin\frac{n\pi x}{L}\,dx.$$

18. Suppose that a laterally insulated rod with length $L = 50$ and thermal diffusivity $k = 1$ has initial temperature $u(x,0) = 0$ and endpoint temperatures $u(0,t) = 0$, $u(50,t) = 100$. Apply the result of Problem 17 to show that

$$u(x,t) = 2x - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left(-n^2\pi^2kt/2500\right)\sin\frac{n\pi x}{50}.$$

19. Suppose that heat is generated within a laterally insulated rod at the rate of $q(x,t)$ calories per second per cubic centimeter. Extend the derivation of the heat equation in this section to derive the equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{q(x,t)}{c\delta}.$$

20. Suppose that current flowing through a laterally insulated rod generates heat at a constant rate; then Problem 19 yields the equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + C.$$

Assume the boundary conditions $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$. **(a)** Find the steady-state temperature $u_{ss}(x)$ determined by

$$0 = k\frac{d^2u_{ss}}{dx^2} + C, \quad u_{ss}(0) = u_{ss}(L) = 0.$$

(b) Show that the transient temperature

$$u_{tr}(x,t) = u(x,t) - u_{ss}(x)$$

satisfies the boundary value problem

$$\frac{\partial u_{\text{tr}}}{\partial t} = k\frac{\partial^2 u_{\text{tr}}}{\partial x^2};$$

$$u_{\text{tr}}(0,t) = u_{\text{tr}}(L,t) = 0,$$

$$u_{\text{tr}}(x,0) = g(x) = f(x) - u_{\text{ss}}(x).$$

Hence conclude from the formulas in (34) and (35) that

$$u(x,t) = u_{\text{ss}}(x) + \sum_{n=1}^{\infty} c_n \exp\left(-n^2\pi^2kt/L^2\right)\sin\frac{n\pi x}{L},$$

where

$$c_n = \frac{2}{L}\int_0^L [f(x) - u_{\text{ss}}(x)]\sin\frac{n\pi x}{L}\,dx.$$

21. The answer to part (a) of Problem 20 is $u_{\text{ss}}(x) = Cx(L-x)/2k$. If $f(x) \equiv 0$ in Problem 20, so the rod being heated is initially at temperature zero, deduce from the result of part (b) that

$$u(x,t) = \frac{Cx}{2k}(L-x) - \frac{4CL^2}{k\pi^3}\sum_{n \text{ odd}} \frac{1}{n^3}\exp\left(-n^2\pi^2kt/L^2\right)\sin\frac{n\pi x}{L}.$$

22. Consider the temperature $u(x,t)$ in a bare slender wire with $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$. Instead of being laterally insulated, the wire loses heat to a surrounding medium (at fixed temperature zero) at a rate proportional to $u(x,t)$. **(a)** Conclude from Problem 19 that

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - hu,$$

where h is a positive constant. **(b)** Then substitute

$$u(x,t) = e^{-ht}v(x,t)$$

to show that $v(x,t)$ satisfies the boundary value problem having the solution given in (30) and (31). Hence conclude that

$$u(x,t) = e^{-ht}\sum_{n=1}^{\infty} c_n \exp\left(-n^2\pi^2kt/L^2\right)\sin\frac{n\pi x}{L},$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

23. Consider a slab with thermal conductivity K occupying the region $0 \leqq x \leqq L$. Suppose that, in accord with Newton's law of cooling, each face of the slab loses heat to the surrounding medium (at temperature zero) at the rate of Hu calories per second per square centimeter. Deduce from Eq. (2) that the temperature $u(x,t)$ in the slab satisfies the boundary conditions

$$hu(0,t) - u_x(0,t) = 0 = hu(L,t) + u_x(L,t)$$

where $h = H/K$.

24. Suppose that a laterally insulated rod with length L, thermal diffusivity k, and initial temperature $u(x,0) = f(x)$ is insulated at the end $x = L$ and held at temperature zero at $x = 0$. **(a)** Separate the variables to show that the eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{2L}$$

for n odd. **(b)** Use the odd half-multiple sine series of Problem 21 in Section 9.3 to derive the solution

$$u(x,t) = \sum_{n \text{ odd}} c_n \exp\left(-n^2\pi^2kt/4L^2\right)\sin\frac{n\pi x}{2L},$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx.$$

9.5 Application Heated-Rod Investigations

First let's investigate numerically the temperature function

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\exp\left(-\frac{n^2\pi^2kt}{L^2}\right)\sin\frac{n\pi x}{L}$$

of the heated rod of Example 2, having length $L = 50$ cm, uniform initial temperature $u_0 = 100$°C, and thermal diffusivity $k = 0.15$ (for iron). The following MATLAB function sums the first N nonzero terms of this series.

```
function u = u(x,t)
k = 0.15;                     % diffusivity of iron
L = 50;                       % length of rod
u0 = 100;                     % initial temperature
S = 0;                        % initial sum
```

```
N = 50;                        % number of terms
for n = 1:2:2*N+1;
     S = S +(1/n)*exp(-n^2*pi^2*k*t/L^2).*sin(n*pi*x/L);
end
u = 4*u0*S/pi;
```

This function was used to plot Figs. 9.5.7 through 9.5.10. The corresponding *Maple* and *Mathematica* functions are provided in the applications manual that accompanies this text. As a practical matter, $N = 50$ terms suffice to give the value $u(x, t)$ after 10 seconds (or longer) with two decimal places of accuracy throughout the interval $0 \leqq x \leqq 50$. (How might you check this assertion?)

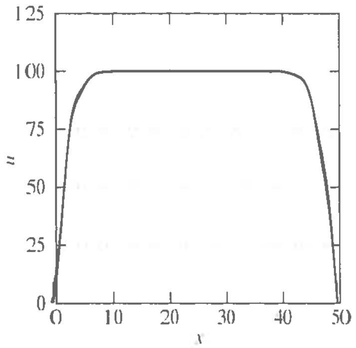


FIGURE 9.5.7. The graph of $u(x, 30)$ giving rod temperatures after 30 seconds.

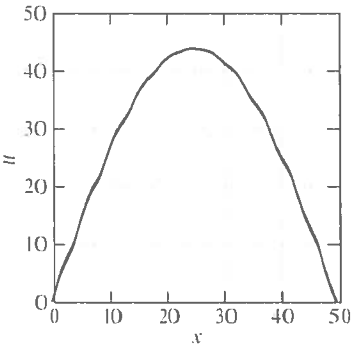


FIGURE 9.5.8. The graph of $u(x, 1800)$ giving rod temperature after 30 minutes.

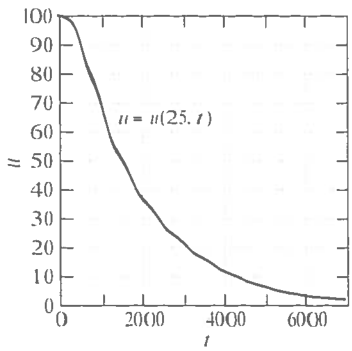


FIGURE 9.5.9. The graph of $u(25, t)$ giving the midpoint temperatures of the rod.

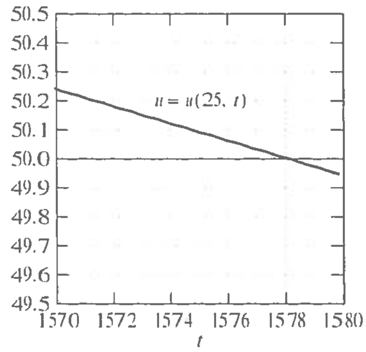


FIGURE 9.5.10. Magnification of the graph of $u(25, t)$ giving the midpoint temperatures of the rod.

The graph of $u(x, 30)$ in Fig. 9.5.7 shows that after 30 seconds the rod has cooled appreciably only near its two ends and still has temperature near 100°C for $10 \leqq x \leqq 40$. Figure 9.5.8 shows the graph of $u(x, 1800)$ after 30 minutes and illustrates the fact (?) that the rod's maximum temperature is always at its midpoint, where $x = 25$.

The graph of $u(25, t)$ for a two-hour period shown in Fig. 9.5.9 indicates that the midpoint temperature takes something more than 1500 seconds (25 minutes) to fall to 50°. Figure 9.5.10 shows a magnification of the graph near its intersection point with the horizontal line $u = 50$ and indicates that this actually takes about 1578 seconds (26 min 18 s).

For your very own rod with constant initial temperature $f(x) = T_0 = 100$ to investigate in this manner, let

$$L = 100 + 10p \quad \text{and} \quad k = 1 + (0.1)q,$$

where p is the largest and q is the smallest nonzero digit of your student ID number.

1. If the two ends of the rod are both held at temperature zero, then determine how long (to the nearest second) it will take for the rod's midpoint temperature to fall to 50°.
2. If the end $x = L$ of the rod is insulated, but the end $x = 0$ is held at temperature zero, then the temperature function $u(x, t)$ is given by the series in Problem 24 of this section. Determine how long it will be until the maximum temperature anywhere in the rod is 50°.

9.6 Vibrating Strings and the One-Dimensional Wave Equation

Although Fourier systematized the method of separation of variables, trigonometric series solutions of partial differential equations had appeared earlier in eighteenth-century investigations of vibrating strings by Euler, d'Alembert, and Daniel Bernoulli. To derive the partial differential equation that models the vibrations of a string, we begin with a flexible uniform string with linear density ρ (in grams per centimeter or slugs per foot) stretched under a tension of T (dynes or pounds) between the fixed points $x = 0$ and $x = L$. Suppose that, as the string vibrates in the xy-plane around its equilibrium position, each point moves parallel to the y-axis, so we can denote by $y(x, t)$ the displacement at time t of the point x of the string. Then, for any fixed value of t, the shape of the string at time t is the curve $y = y(x, t)$. We assume also that the deflection of the string remains so slight that the approximation $\sin\theta \approx \tan\theta = y_x(x, t)$ is quite accurate (Fig. 9.6.1). Finally, we assume that in addition to the internal forces of tension acting tangentially to the string, it is acted on by an external vertical force with linear density $F(x)$ in such units as dynes per centimeter or pounds per foot.

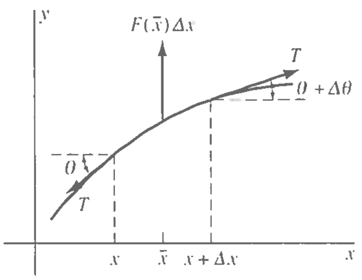


FIGURE 9.6.1. Forces on a short segment of the vibrating string.

We want to apply Newton's second law $F = ma$ to the short segment of string of mass $\rho\,\Delta x$ corresponding to the interval $[x, x + \Delta x]$, with a being the vertical acceleration $y_{tt}(\bar{x}, t)$ of its midpoint. Reading the vertical components of the force shown in Fig. 9.6.1, we get

$$\begin{aligned}(\rho\,\Delta x) \cdot y_{tt}(\bar{x}, t) &\approx T\sin(\theta + \Delta\theta) - T\sin\theta + F(\bar{x})\,\Delta x \\ &\approx T y_x(x + \Delta x, t) - T y_x(x, t) + F(\bar{x})\,\Delta x,\end{aligned}$$

so division by Δx yields

$$\rho y_{tt}(\bar{x}, t) \approx T\frac{y_x(x + \Delta x, t) - y_x(x, t)}{\Delta x} + F(\bar{x}).$$

We now take limits in this equation as $\Delta x \to 0$, so $\bar{x} \to x$ (because $\bar{x}$ lies in the interval $[x, x + \Delta x]$ with fixed left endpoint x). Then the two sides of the equation approach the two sides of the partial differential equation

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} + F(x) \tag{1}$$

that describes the vertical vibrations of a flexible string with constant linear density ρ and tension T under the influence of an external vertical force with linear density $F(x)$.

If we set

$$a^2 = \frac{T}{\rho} \tag{2}$$

and set $F(x) \equiv 0$ in Eq. (1), we get the **one-dimensional wave equation**

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{3}$$

that models the *free* vibrations of a uniform flexible string.

The fixed ends of the string at the points $x = 0$ and $x = L$ on the x-axis correspond to the *endpoint conditions*

$$y(0, t) = y(L, t) = 0. \tag{4}$$

Our intuition about the physics of the situation suggests that the motion of the string will be determined if we specify both its **initial position function**

$$y(x, 0) = f(x) \qquad (0 < x < L) \tag{5}$$

and its **initial velocity function**

$$y_t(x, 0) = g(x) \qquad (0 < x < L). \tag{6}$$

On combining Eqs. (3) through (6), we get the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0); \tag{7a}$$

$$y(0, t) = y(L, t) = 0, \tag{7b}$$

$$y(x, 0) = f(x) \qquad (0 < x < L), \tag{7c}$$

$$y_t(x, 0) = g(x) \qquad (0 < x < L) \tag{7d}$$

for the displacement function $y(x, t)$ of a freely vibrating string with fixed ends, initial position $f(x)$, and initial velocity $g(x)$.

Solution by Separation of Variables

Like the heat equation, the wave equation in (7a) is linear: Any linear combination of two solutions is again a solution. Another similarity is that the endpoint conditions in (7b) are homogeneous. Unfortunately, the conditions in both (7c) and (7d) are nonhomogeneous; we must deal with *two* nonhomogeneous boundary conditions.

As we described in Section 9.5, the method of separation of variables involves superposition of solutions satisfying the homogeneous conditions to obtain a solution that also satisfies a single nonhomogeneous boundary condition. To adapt the technique to the situation at hand, we adopt the "divide and conquer" strategy of splitting the problem in (7) into the following two separate boundary value problems, each involving only a single nonhomogeneous boundary condition:

Problem A	Problem B
$y_{tt} = a^2 y_{xx}$;	$y_{tt} = a^2 y_{xx}$;
$y(0, t) = y(L, t) = 0$,	$y(0, t) = y(L, t) = 0$,
$y(x, 0) = f(x)$,	$y(x, 0) = 0$,
$y_t(x, 0) = 0$.	$y_t(x, 0) = g(x)$.

If we can separately find a solution $y_A(x, t)$ of Problem A and a solution $y_B(x, t)$ of Problem B, then their sum $y(x, t) = y_A(x, t) + y_B(x, t)$ will be a solution of the original problem in (7), because

$$y(x, 0) = y_A(x, 0) + y_B(x, 0) = f(x) + 0 = f(x)$$

and

$$y_t(x, 0) = \{y_A\}_t(x, 0) + \{y_B\}_t(x, 0) = 0 + g(x) = g(x).$$

So let us attack Problem A with the method of separation of variables. Substitution of

$$y(x, t) = X(x)T(t) \tag{8}$$

in $y_{tt} = a^2 y_{xx}$ yields $XT'' = a^2 X''T$ where (as before) we write X'' for $X''(x)$ and T'' for $T''(t)$. Therefore,

$$\frac{X''}{X} = \frac{T''}{a^2 T}. \tag{9}$$

The functions X''/X of x and T''/a^2T of t can agree for all x and t only if each is equal to the same constant. Consequently, we may conclude that

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda \tag{10}$$

for some constant λ; the minus sign is inserted here merely to facilitate recognition of the eigenvalue problem in (13). Thus our partial differential equation separates into the two ordinary differential equations

$$X'' + \lambda X = 0, \tag{11}$$

$$T'' + \lambda a^2 T = 0. \tag{12}$$

The endpoint conditions

$$y(0, t) = X(0)T(t) = 0, \quad y(L, t) = X(L)T(t) = 0$$

require that $X(0) = X(L) = 0$ if $T(t)$ is nontrivial. Hence $X(x)$ must satisfy the now familiar eigenvalue problem

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0. \tag{13}$$

As in Eqs. (23) and (24) of Section 9.5, the eigenvalues of this problem are the numbers

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots, \tag{14}$$

and the eigenfunction associated with λ_n is

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots. \tag{15}$$

Now we turn to Eq. (12). The homogeneous initial condition

$$y_t(x, 0) = X(x)T'(0) = 0$$

implies that $T'(0) = 0$. Therefore, the solution $T_n(t)$ associated with the eigenvalue $\lambda_n = n^2\pi^2/L^2$ must satisfy the conditions

$$T_n'' + \frac{n^2\pi^2a^2}{L^2}T_n = 0, \quad T_n'(0) = 0. \tag{16}$$

The general solution of the differential equation in (16) is

$$T_n(t) = A_n\cos\frac{n\pi at}{L} + B_n\sin\frac{n\pi at}{L}. \tag{17}$$

Its derivative

$$T_n'(t) = \frac{n\pi a}{L}\left(-A_n\sin\frac{n\pi at}{L} + B_n\cos\frac{n\pi at}{L}\right)$$

satisfies the condition $T_n'(0) = 0$ if $B_n = 0$. Thus a nontrivial solution of (16) is

$$T_n(t) = \cos\frac{n\pi at}{L}. \tag{18}$$

We combine the results in Eqs. (15) and (18) to obtain the infinite sequence of product functions

$$y_n(x, t) = X_n(x)T_n(t) = \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}, \tag{19}$$

$n = 1, 2, 3, \ldots$. Each of these *building block* functions satisfies both the wave equation $y_{tt} = a^2y_{xx}$ and the homogeneous boundary conditions in Problem A. By superposition we get the infinite series

$$y_n(x, t) = \sum_{n=1}^{\infty} A_nX_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}. \tag{20}$$

It remains only to choose the coefficients $\{A_n\}$ to satisfy the nonhomogeneous boundary condition

$$y(x, 0) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L} = f(x) \tag{21}$$

for $0 < x < L$. But this will be the Fourier sine series of $f(x)$ on $[0, L]$ provided that we choose

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{22}$$

Thus we see finally that a formal series solution of Problem A is

$$y_A(x, t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi at}{L} \sin\frac{n\pi x}{L}, \tag{23}$$

with the coefficients $\{A_n\}_1^\infty$ computed using Eq. (22). Note that the series in (23) is obtained from the Fourier sine series of $f(x)$ simply by inserting the factor $\cos(n\pi at/L)$ in the nth term. Note also that this term has (circular) frequency $\omega_n = n\pi a/L$.

Example 1 It follows immediately that the solution of the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2} \qquad (0 < x < \pi,\ t > 0);$$
$$y(0, t) = y(\pi, t) = 0,$$
$$y(x, 0) = \tfrac{1}{10}\sin^3 x = \tfrac{3}{40}\sin x - \tfrac{1}{40}\sin 3x,$$
$$y_t(x, 0) = 0,$$

for which $L = \pi$ and $a = 2$, is

$$y(x, t) = \tfrac{3}{40}\cos 2t\ \sin x - \tfrac{1}{40}\cos 6t\ \sin 3x.$$

The reason is that we are given explicitly the Fourier sine series of $f(x)$ with $A_1 = \frac{3}{40}$, $A_3 = -\frac{1}{40}$, and $A_n = 0$ otherwise. ■

Example 2 **A plucked string** Figure 9.6.2 shows the initial position function $f(x)$ for a stretched string (of length L) that is set in motion by moving its midpoint $x = L/2$ aside the distance $\frac{1}{2}bL$ and then releasing it from rest at time $t = 0$. The corresponding boundary value problem is

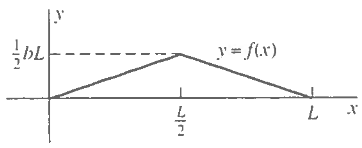


FIGURE 9.6.2. The initial position of the plucked string of Example 2.

$$y_{tt} = a^2 y_{xx} \qquad (0 < x < L,\ t > 0);$$
$$y(0, t) = y(L, t) = 0,$$
$$y(x, 0) = f(x),$$
$$y_t(x, 0) = 0,$$

where $f(x) = bx$ for $0 \leqq x \leqq L/2$ and $f(x) = b(L - x)$ for $L/2 \leqq x \leqq L$. Find $y(x, t)$.

Solution The nth Fourier sine coefficient of $f(x)$ is

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{L}\int_0^{L/2} bx\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L} b(L-x)\sin\frac{n\pi x}{L}\,dx;$$

it follows that

$$A_n = \frac{4bL}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Hence Eq. (23) yields the formal series solution

$$\begin{aligned} y(x, t) &= \frac{4bL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L} \\ &= \frac{4bL}{\pi^2} \left(\cos \frac{\pi a t}{L} \sin \frac{\pi x}{L} - \frac{1}{3^2} \cos \frac{3\pi a t}{L} \sin \frac{3\pi x}{L} + \cdots \right). \end{aligned} \tag{24}$$

■

Music

Numerous familiar musical instruments employ vibrating strings to generate the sounds they produce. When a string vibrates with a given frequency, vibrations at this frequency are transmitted through the air—in the form of periodic variations in air density called **sound waves**—to the ear of the listener. For example, middle C is a *tone* with a frequency of approximately 256 Hz. When several tones are heard simultaneously, the combination is perceived as harmonious if the ratios of their frequencies are nearly ratios of small whole numbers; otherwise many perceive the combinations as dissonant.

The series in Eq. (23) represents the motion of a string as a superposition of infinitely many vibrations with different frequencies. The nth term

$$A_n \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$$

represents a vibration with frequency

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{n\pi a/L}{2\pi} = \frac{n}{2L}\sqrt{\frac{T}{\rho}} \quad \text{(Hz)}. \tag{25}$$

The lowest of these frequencies,

$$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \quad \text{(Hz)}, \tag{26}$$

is called the **fundamental frequency,** and it is ordinarily predominant in the sound we hear. The frequency $\nu_n = n\nu_1$ of the nth **overtone** or **harmonic** is an integral multiple of ν_1, and this is why the sound of a single vibrating string is harmonious rather than dissonant.

Note in Eq. (26) that the fundamental frequency ν_1 is proportional to $\sqrt{T}$ and inversely proportional to L and to $\sqrt{\rho}$. Thus we can double this frequency—and hence get a fundamental tone one **octave** higher—either by halving the length L or by quadrupling the tension T. The initial conditions do *not* affect ν_1; instead, they determine the coefficients in (23) and hence the extent to which the higher harmonics contribute to the sound produced. Therefore the initial conditions affect the **timbre**, or overall frequency mixture, rather than the fundamental frequency. (Technically this is true only for relatively small displacements of the string; if you strike a piano key rather forcefully you can detect a slight and brief initial deviation from the usual frequency of the note.)

According to one (rather simplistic) theory of hearing, the loudness of the sound produced by a vibrating string is proportional to its total (kinetic plus potential) energy, which is given by

$$E = \frac{1}{2}\int_0^L \left[\rho \left(\frac{\partial y}{\partial t}\right)^2 + T \left(\frac{\partial y}{\partial x}\right)^2 \right] dx. \tag{27}$$

reached. But then $y(x,0) = X(x)T(0) = 0$ implies that $T(0) = 0$, so instead of (16) we have

$$\frac{d^2T_n}{dt^2} + \frac{n^2\pi^2a^2}{L^2}T_n = 0, \quad T_n(0) = 0. \tag{31}$$

From Eq. (17) we see that a nontrivial solution of (31) is

$$T_n(t) = \sin\frac{n\pi at}{L}. \tag{32}$$

The resulting formal power series solution is therefore of the form

$$y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi at}{L} \sin\frac{n\pi x}{L}, \tag{33}$$

so we want to choose the coefficients $\{B_n\}$ so that

$$y_t(x,0) = \sum_{n=1}^{\infty} B_n\frac{n\pi a}{L} \sin\frac{n\pi x}{L} = g(x). \tag{34}$$

Thus we want $B_n \cdot n\pi a/L$ to be the Fourier sine coefficient b_n of $g(x)$ on $[0, L]$:

$$B_n\frac{n\pi a}{L} = b_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Hence we choose

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \tag{35}$$

in order for $y_B(x,t)$ in (33) to be a formal series solution of Problem B—and thus for $y(x,t) = y_A(x,t) + y_B(x,t)$ to be a formal series solution of our original boundary value problem in Eqs. (7a)–(7d).

Example 3 Consider a string on a guitar lying crosswise in the back of a pickup truck that at time $t = 0$ slams into a brick wall with speed v_0. Then $g(x) \equiv v_0$, so

$$B_n = \frac{2}{n\pi a}\int_0^L v_0 \sin\frac{n\pi x}{L}\,dx = \frac{2v_0L}{n^2\pi^2a}\left[1 - (-1)^n\right].$$

Hence the series in (33) gives

$$y(x,t) = \frac{4v_0L}{\pi^2a}\sum_{n\text{ odd}}\frac{1}{n^2}\sin\frac{n\pi at}{L}\sin\frac{n\pi x}{L}.$$ ■

If we differentiate the series in (33) termwise with respect to t, we get

$$y_t(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L} = \tfrac{1}{2}[G(x+at) + G(x-at)]. \tag{36}$$

where G is the odd period $2L$ extension of the initial velocity function $g(x)$, using the same device as in the derivation of Eq. (30). In Problem 15 we ask you to deduce that

$$y(x,t) = \frac{1}{2a}\left[H(x+at) + H(x-at)\right], \tag{37}$$

where the function $H(x)$ is defined to be

$$H(x) = \int_0^x G(s)\,ds. \tag{38}$$

If, finally, a string has both a nonzero initial position function $y(x,0) = f(x)$ and a nonzero initial velocity function $y_t(x,0) = g(x)$, then we can obtain its displacement function by adding the d'Alembert solutions of Problems A and B given in Eqs. (30) and (37), respectively. Hence the vibrations of this string with general initial conditions are described by

$$y(x,t) = \frac{1}{2}\left[F(x+at) + F(x-at)\right] + \frac{1}{2a}\left[H(x+at) + H(x-at)\right], \tag{39}$$

a superposition of four waves moving along the x-axis with speed a, two moving to the left and two to the right.

9.6 Problems

've the boundary value problems in Problems 1 through 10.

. $y_{tt} = 4y_{xx}$, $0 < x < \pi$, $t > 0$; $y(0,t) = y(\pi,t) = 0$, $y(x,0) = \frac{1}{10}\sin 2x$, $y_t(x,0) = 0$

. $y_{tt} = y_{xx}$, $0 < x < 1$, $t > 0$; $y(0,t) = y(1,t) = 0$, $y(x,0) = \frac{1}{10}\sin \pi x - \frac{1}{20}\sin 3\pi x$, $y_t(x,0) = 0$

. $4y_{tt} = y_{xx}$, $0 < x < \pi$, $t > 0$; $y(0,t) = y(\pi,t) = 0$, $y(x,0) = y_t(x,0) = \frac{1}{10}\sin x$

. $4y_{tt} = y_{xx}$, $0 < x < 2$, $t > 0$; $y(0,t) = y(2,t) = 0$, $y(x,0) = \frac{1}{5}\sin \pi x \cos \pi x$, $y_t(x,0) = 0$

. $y_{tt} = 25y_{xx}$, $0 < x < 3$, $t > 0$; $y(0,t) = y(3,t) = 0$, $y(x,0) = \frac{1}{4}\sin \pi x$, $y_t(x,0) = 10\sin 2\pi x$

. $y_{tt} = 100y_{xx}$, $0 < x < \pi$, $t > 0$; $y(0,t) = y(\pi,t) = 0$, $y(x,0) = x(\pi - x)$, $y_t(x,0) = 0$

. $y_{tt} = 100y_{xx}$, $0 < x < 1$, $t > 0$; $y(0,t) = y(1,t) = 0$, $y(x,0) = 0$, $y_t(x,0) = x$

. $y_{tt} = 4y_{xx}$, $0 < x < \pi$, $t > 0$; $y(0,t) = y(\pi,t) = 0$, $y(x,0) = \sin x$, $y_t(x,0) = 1$

. $y_{tt} = 4y_{xx}$, $0 < x < 1$, $t > 0$; $y(0,t) = y(1,t) = 0$, $y(x,0) = 0$, $y_t(x,0) = x(1-x)$

. $y_{tt} = 25y_{xx}$, $0 < x < \pi$, $t > 0$; $y(0,t) = y(\pi,t) = 0$, $y(x,0) = y_t(x,0) = \sin^2 x$

. Suppose that a string 2 ft long weighs $\frac{1}{32}$ oz and is subjected to a tension of 32 lb. Find the fundamental frequency with which it vibrates and the velocity with which the vibration waves travel along it.

. Show that the amplitude of the oscillations of the midpoint of the string of Example 3 is

$$y\left(\frac{L}{2}, \frac{L}{2a}\right) = \frac{4v_0 L}{\pi^2 a}\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{v_0 L}{2a}.$$

If the string is the string of Problem 11 and the impact speed of the pickup truck is 60 mi/h, show that this amplitude is approximately 1 in.

13. Suppose that the function $F(x)$ is twice differentiable for all x. Use the chain rule to verify that the functions

$$y(x,t) = F(x+at) \quad \text{and} \quad y(x,t) = F(x-at)$$

satisfy the equation $y_{tt} = a^2 y_{xx}$.

14. Given the differentiable odd period $2L$ function $F(x)$, show that the function

$$y(x,t) = \tfrac{1}{2}\left[F(x+at) + F(x-at)\right]$$

satisfies the conditions $y(0,t) = y(L,t) = 0$, $y(x,0) = F(x)$, and $y_t(x,0) = 0$.

15. If $y(x,0) = 0$, then Eq. (36) implies (why?) that

$$y(x,t) = \tfrac{1}{2}\int_0^t G(x+a\tau)\,d\tau + \tfrac{1}{2}\int_0^t G(x-a\tau)\,d\tau.$$

Make appropriate substitutions in these integrals to derive Eqs. (37) and (38).

16. (a) Show that the substitutions $u = x + at$ and $v = x - at$ transform the equation $y_{tt} = a^2 y_{xx}$ into the equation $y_{uv} = 0$. (b) Conclude that every solution of $y_{tt} = a^2 y_{xx}$ is of the form

$$y(x, t) = F(x + at) + G(x - at),$$

which represents two waves traveling in opposite directions, each with speed a.

17. Suppose that

$$y(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L}.$$

Square the derivatives y_t and y_x and then integrate termwise—applying the orthogonality of the sine and cosine functions—to verify that

$$E = \frac{1}{2}\int_0^L (\rho y_t^2 + T y_x^2)\,dx = \frac{\pi^2 T}{4L}\sum_{n=1}^{\infty} n^2 A_n^2.$$

18. Consider a stretched string, initially at rest; its end at $x = 0$ is fixed, but its end at $x = L$ is partially free—it is allowed to slide without friction along the vertical line $x = L$. The corresponding boundary value problem is

$$\begin{aligned} &y_{tt} = a^2 y_{xx} \qquad (0 < x < L,\ t > 0); \\ &y(0, t) = y_x(L, t) = 0, \\ &y(x, 0) = f(x), \\ &y_t(x, 0) = 0. \end{aligned}$$

Separate the variables and use the odd half-multiple sine series of $f(x)$, as in Problem 24 of Section 9.5, to derive the solution

$$y(x, t) = \sum_{n \text{ odd}} A_n \cos \frac{n\pi at}{2L} \sin \frac{n\pi x}{2L},$$

where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{2L}\,dx.$$

Problems 19 and 20 deal with the vibrations of a string under the influence of the downward force $F(x) = -\rho g$ of gravity. According to Eq. (1), its displacement function satisfies the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - g \tag{40}$$

with endpoint conditions $y(0, t) = y(L, t) = 0$.

19. Suppose first that the string hangs in a stationary position, so that $y = y(x)$ and $y_{tt} = 0$, and hence its differential equation of motion takes the simple form $a^2 y'' = g$. Derive the stationary solution

$$y(x) = \phi(x) = \frac{gx}{2a^2}(x - L).$$

20. Now suppose that the string is released from rest in equilibrium; consequently the initial conditions are $y(x, 0) = 0$ and $y_t(x, 0) = 0$. Define

$$v(x, t) = y(x, t) - \phi(x),$$

where $\phi(x)$ is the stationary solution of Problem 19. Deduce from Eq. (40) that $v(x, t)$ satisfies the boundary value problem

$$\begin{aligned} &v_{tt} = a^2 v_{xx}; \\ &v(0, t) = v(L, t) = 0, \\ &v(x, 0) = -\phi(x), \\ &v_t(x, 0) = 0. \end{aligned}$$

Conclude from Eqs. (22) and (23) that

$$y(x, t) - \phi(x) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L},$$

where the coefficients $\{A_n\}$ are the Fourier sine coefficients of $f(x) = -\phi(x)$. Finally, explain why it follows that the string oscillates between the positions $y = 0$ and $y = 2\phi(x)$.

21. For a string vibrating in air with resistance proportional to velocity, the boundary value problem is

$$\begin{aligned} &y_{tt} = a^2 y_{xx} - 2h y_t; \\ &y(0, t) = y(L, t) = 0, \\ &y(x, 0) = f(x), \\ &y_t(x, 0) = 0. \end{aligned} \tag{41}$$

Assume that $0 < h < \pi a/L$. (a) Substitute

$$y(x, t) = X(x)T(t)$$

in (41) to obtain the equations

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0 \tag{42}$$

and

$$T'' + 2hT' + a^2\lambda T = 0, \quad T'(0) = 0. \tag{43}$$

(b) The eigenvalues and eigenfunctions of (42) are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin \frac{n\pi x}{L}$$

(as usual). Show that the general solution with $\lambda = n^2\pi^2/L^2$ of the differential equation in (43) is

$$T_n(t) = e^{-ht}(A_n \cos \omega_n t + B_n \sin \omega_n t),$$

where $\omega_n = \sqrt{(n^2\pi^2a^2/L^2) - h^2} < n\pi a/L$. (c) Show that $T_n'(0) = 0$ implies that $B_n = hA_n/\omega_n$, and hence that to within a constant multiplicative coefficient,

$$T_n(t) = e^{-ht} \cos (\omega_n t - \alpha_n)$$

where $\alpha_n = \tan^{-1}(h/\omega_n)$. (d) Finally, conclude that

$$y(x, t) = e^{-ht} \sum_{n=1}^{\infty} c_n \cos (\omega_n t - \alpha_n) \sin \frac{n\pi x}{L}.$$

where

$$c_n = \frac{2}{L\cos\alpha_n}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

From this formula we see that the air resistance has three main effects: exponential damping of amplitudes, decreased frequencies $\omega_n < n\pi a/L$, and the introduction of the phase delay angles α_n.

22. Rework Problem 21 as follows: First substitute $y(x, t) = e^{-ht}v(x, t)$ in Eq. (41) and then show that the boundary value problem for $v(x, t)$ is

$$\begin{aligned} v_{tt} &= a^2 v_{xx} + h^2 v; \\ v(0, t) &= v(L, t) = 0, \\ v(x, 0) &= f(x), \\ v_t(x, 0) &= hf(x). \end{aligned}$$

Next show that the substitution $v(x, t) = X(x)T(t)$ leads to the equations

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0,$$
$$T'' + (\lambda a^2 - h^2)T = 0.$$

Proceed in this manner to derive the solution $y(x, t)$ given in part (d) of Problem 21.

The snapshots in Fig. 9.6.4 show successive positions of a vibrating string with length $L = \pi$ and $a = 1$ (so its period of oscillation is 2π). The string is initially at rest with fixed endpoints, and at time $t = 0$ it is set in motion with initial position function

$$f(x) = 2\sin^2 x = 1 - \cos 2x. \qquad (44)$$

23. The most interesting snapshot is the one in Fig. 9.6.4(e), where it appears that the string exhibits a momentary "flat spot" at the instant $t = \pi/4$. Indeed, apply the d'Alembert formula in (30) to prove that the string's position function $y(x, t)$ satisfies the condition

$$y\left(x, \frac{\pi}{4}\right) = 1 \quad \text{for} \quad \frac{\pi}{4} \leqq x \leqq \frac{3\pi}{4}.$$

24. **(a)** Show that the position function $f(x)$ defined in Eq. (44) has inflection points $[f''(x) = 0]$ at $x = \pi/4$ and at $x = 3\pi/4$. **(b)** In snapshots (a)–(c) of Fig. 9.6.4 it appears that these two inflection points may remain fixed during some initial portion of the string's vibration. Indeed, apply the d'Alembert formula to show that if either $x = \pi/4$ or $x = 3\pi/4$, then $y(x, t) = 1$ for $0 \leqq t \leqq \pi/4$.

9.6 Application Vibrating-String Investigations

Here we describe a *Mathematica* implementation of the d'Alembert solution

$$y(x, t) = \tfrac{1}{2}\left[F(x + at) + F(x - at)\right] \qquad (1)$$

of the vibrating-string problem, and apply it to investigate graphically the motions that result from a variety of different initial positions of the string. *Maple* and MATLAB versions of this implementation are included in the applications manual that accompanies this text.

To plot the snapshots shown in Fig. 9.6.4, we began with the initial position function

```
f[x_] := 2*Sin[x]^2
```

To define the odd period 2π extension $F(x)$ of $f(x)$, we need the function $s(x)$ that shifts the point x by an integral multiple of π into the interval $[0, \pi]$.

```
s[x_] := Block[{k}, k = Floor[N[x/Pi]];
            If[EvenQ[k], (* k is even *)
            (* then *) N[x - k*Pi],
            (* else *) N[x - k*Pi - Pi]]]
```

Then the desired odd extension is defined by

```
F[x_] := If[s[x] > 0, (* then *) f[s[x]],
            (* else *)-f[-s[x]]]
```

Finally, the d'Alembert solution in (1) is

```
G[x_, t_] := ( F[x + t] + F[x - t] )/2
```