Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)
   (a) [40%] Solve \( y' = \frac{3 + x^2}{2 + x} \).

   (b) [60%] Find the position \( x(t) \) from the velocity model \( \frac{d}{dt}(e^t v(t)) = 2 e^{2t}, \ v(0) = 5 \) and the position model \( \frac{dx}{dt} = v(t) \), \( x(2) = 2 \).

Solution to Problem 1.
(a) Answer \( y(x) = \frac{1}{2} x^2 - 2x + 7 \ln(2 + x) + c \). The integral of \( F(x) = \frac{3 + x^2}{2 + x} \) is found by substitution \( u = 2 + x \), resulting in the new integration problem \( \int F(x) dx = \int (u - 4 + 7/u) du \).

(b) Velocity \( v(t) = e^t + 4e^{-t} \) by quadrature. Integrate \( x'(t) = e^t + 4e^{-t} \) with \( x(0) = 2 \) to obtain position \( x(t) = e^t - 4e^{-t} + 5 \).

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2. (Classification of Equations)

The differential equation \( y' = f(x, y) \) is defined to be **separable** provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

(a) [40%] The equation \( y' + x(y + 3) = ye^x + 3x \) is separable. Provide formulas for \( F(x) \) and \( G(y) \).

(b) [60%] Apply partial derivative tests to show that \( y' = x + y \) is linear but not separable. Supply all details.

**Solution to Problem 2.**

(a) The equation is \( y' = ye^x - xy = (e^x - x)y \). Then \( F(x) = e^x - x \), \( G(y) = y \).

(b) Let \( f(x, y) = x + y \). Then \( \partial f/\partial y = 1 \), which is independent of \( y \), hence the equation \( y' = f(x, y) \) is linear. The negative test is \( \frac{\partial f/\partial x}{f} \) depends on \( x \). In this case, the fraction is

\[
\frac{\partial f/\partial x}{f} = \frac{1}{f} = \frac{1}{x + y}.
\]

At \( y = 0 \), this reduces to \( 1/x \), which depends on \( x \), therefore the equation \( y' = f(x, y) \) is not separable.

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3. (Solve a Separable Equation)
Given \((5y + 10)y' = (xe^{-x} + \sin(x) \cos(x)) (y^2 + 3y - 4)\).

Find a non-equilibrium solution in implicit form.
To save time, do not solve for \(y\) explicitly and do not solve for equilibrium solutions.

**Solution to Problem 3.**
The solution by separation of variables identifies the separated equation \( y' = F(x)G(y) \) using
\[
F(x) = xe^{-x} + \sin(x) \cos(x), \quad G(y) = \frac{y^2 + 3y - 4}{5y + 10}.
\]
The integral of \(F\) is done by parts and also by u-substitution.
\[
\int F dx = \int xe^{-x} dx + \int \sin(x) \cos(x) dx
= I_1 + I_2.
\]
\[
I_1 = \int xe^{-x} dx
= -xe^{-x} - \int e^{-x} dx, \quad \text{parts } u = x, dv = e^{-x} dx,
= xe^{-x} - e^{-x} + c_1
\]
\[
I_2 = \int \sin(x) \cos(x) dx
= \int u du, \quad u = \sin(x), du = \cos(x) dx,
= \frac{u^2}{2} + c_2
= \frac{1}{2} \sin^2(x) + c_2
\]
Then \(\int F dx = xe^{-x} - e^{-x} + \frac{1}{2} \sin^2(x) + c_3\).

The integral of \(1/G(y)\) requires partial fractions. The details:
\[
\int \frac{dx}{G(y(x))} = \int \frac{5u + 10}{u^2 + 3u - 4} du, \quad u = y(x), du = y'(x) dx,
= \int \frac{5u + 10}{(u + 4)(u - 1)} du
= \int \frac{A}{u + 4} + \frac{B}{u - 1} du, \quad A, B \text{ determined later},
= A \ln |u + 4| + B \ln |u - 1| + c_4
\]
The partial fraction problem
\[
\frac{5u + 10}{(u + 4)(u - 1)} = \frac{A}{u + 4} + \frac{B}{u - 1}
\]
can be solved in a variety of ways, with answer \(A = \frac{-20 + 10}{5} = 2\) and \(B = \frac{15}{5} = 3\). The final implicit solution is obtained from \(\int \frac{dx}{G(y(x))} = \int F(x) dx\), which gives the equation
\[
2 \ln |y + 4| + 3 \ln |y - 1| = xe^{-x} - e^{-x} + \frac{1}{2} \sin^2(x) + c.
\]

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4. (Linear Equations)

(a) [60%] Solve the linear model \(2x'(t) = -64 + \frac{10}{3t + 2}x(t)\), \(x(0) = 32\). Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation \(\frac{dy}{dx} - (\cos(x))y = 0\).

(c) [20%] Solve \(5\frac{dy}{dx} - 7y = 10\) using the superposition principle \(y = y_h + y_p\). Expected are answers for \(y_h\) and \(y_p\).

Solution to Problem 4.

(a) The answer is \(v(t) = 32 + 48t\). The details:

\[
v'(t) = -32 + \frac{5}{3t + 2}v(t),
\]

\[
v'(t) + \frac{-5}{3t + 2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t)
\]

\[
p(t) = -\frac{5}{3t + 2}, \quad \text{integrating factor}
\]

\[
W = e^{\int p \, dt}, \quad \text{integrating factor}
\]

\[
W = e^n, \quad u = \int p \, dt = -\frac{5}{3}\ln |3t + 2| = \ln |3t + 2|^{-5/3}
\]

\[
W = (3t + 2)^{-5/3}, \quad \text{Final choice for } W.
\]

Then replace the left side of \(v' + pv = q\) by \((vW)'W\) to obtain

\[
(vW)' = -32, \quad \text{Replace left side by quotient } (vW)'W
\]

\[
(vW)' = -32W, \quad \text{cross-multiply}
\]

\[
vW = -32 \int W \, dt, \quad \text{quadrature step.}
\]

The evaluation of the integral is from the power rule:

\[
\int -32W \, dt = -32 \int (3t + 2)^{-5/3} \, dt = -32 \frac{(3t + 2)^{-2/3}}{(-2/3)(3)} + c.
\]

Division by \(W = (3t + 2)^{-5/3}\) then gives the general solution

\[
v(t) = \frac{c}{W} - \frac{32}{2}(3t + 2)^{-2/3}(3t + 2)^{5/3}.
\]

Constant \(c\) evaluates to \(c = 0\) because of initial condition \(v(0) = 32\). Then

\[
v(t) = \frac{32}{2}(3t + 2)^{-2/3}(3t + 2)^{5/3} = 16(3t + 2)^{-\frac{2}{3} + \frac{5}{3}} = 16(3t + 2).
\]

(b) The answer is \(y = \text{constant divided by the integrating factor: } y = \frac{c}{W}\). Because \(W = e^n\) where \(u = \int -\cos(x) \, dx = -\sin x\), then \(y = ce^{\sin x}\).

(c) The equilibrium solution (a constant solution) is \(y_p = -\frac{10}{7}\). The homogeneous solution is \(y_h = ce^{\frac{7x}{5}}\) = constant divided by the integrating factor. Then \(y = y_p + y_h = -\frac{10}{7} + ce^{\frac{7x}{5}}\).

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5. (Stability)

Assume an autonomous equation \( x'(t) = f(x(t)) \). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

\[
\begin{array}{cccccc}
+ & - & + & - & + & + \\
-10 & -5 & -3 & 0 & 3 & x
\end{array}
\]

Solution to Problem 5.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point.
5. Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:
\[ x = -10: \text{FUNNEL, STABLE} \]
\[ x = -5: \text{SPOUT, UNSTABLE} \]
\[ x = -3: \text{FUNNEL, STABLE} \]
\[ x = 0: \text{SPOUT, UNSTABLE} \]
\[ x = 3: \text{NODE, STABLE} \]

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6. (ch3)
Using Euler’s theorem on atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a differential equation
\[ ay'' + by' + cy = 0 \]
which has particular solutions
\[-5e^{-x} + xe^{-x}, 10e^{-x} + xe^{-x}.\]

(b) [30%] Given characteristic equation
\[ r(r - 2)(r^3 + 4r^2 + 2r + 17) = 0, \]
solve the differential equation.

(c) [30%] Given
\[ mx''(t) + cx'(t) + kx(t) = 0, \]
which represents an unforced damped spring-mass system. Assume
\[ m = 4, c = 4, k = 129. \]
Classify the answer as over-damped, critically damped or under-damped. Illustrate in a drawing the assignment of physical constants \[ m, c, k \] and the initial conditions \[ x(0) = 0, x'(0) = 1. \]

Solution to Problem 6.

6(a)
Divide the first solution by 2. Then Euler atom \( e^{-x} \) is a solution, which implies that \( r = -1 \) is a root of the characteristic equation. Subtract \( y_1 = e^{-x} \) and \( y_2 = e^{-x} - e^{2x/3} \) to justify that \( y = y_1 - y_2 = e^{2x/3} \) is a solution. It is an Euler atom corresponding to root \( r = 2/3 \). Then the characteristic equation should be \( (r - (-1))(r - 2/3) = 0, \) or \( 3r^2 + r - 2 = 0. \) The differential equation is
\[ 3y'' + y' - 2y = 0. \]

6(b)
The characteristic equation factors into \( r^4(r^2 + 4r + 4) = 0 \) with roots \( r = 0, 0, 0, 0, -2, -2. \) Then \( y \) is a linear combination of the Euler atoms \( 1, x, x^2, x^3, e^{-2x}, xe^{-2x}. \)

6(c)
The roots of the fully factored equation \( r^4(r + 2)^4(r - 2)^3((r + 1)^2 + 4) = 0 \) are
\[ r = 0, 0, 0, 0, -2, -2, -2, 2, 2, 2, -1 \pm 2i. \]
The solution \( y \) is a linear combination of the Euler atoms
\[ 1, x, x^2, x^3; \]
\[ e^{-2x}, xe^{-2x}, e^{-2x} + e^{2x}, xe^{2x}, xe^{-2x} + e^{2x}; \]
\[ e^{-x} \cos(2x), e^{-x} \sin(2x), \]
\[ e^{-x} \cos(2x), e^{-x} \sin(2x). \]

6(d)
Use \( 4r^2 + 4r + 65 = 0 \) and the quadratic formula to obtain roots \( r = -1/2 + 4i, -1/2 - 4i \) and Euler atoms \( \cos 4t, \sin 4t. \) Then \( y = (c_1 \cos 4t + c_2 \sin 4t) e^{-1/2}. \) This is under-damped (it oscillates). The illustration shows a spring, a dashpot and a mass with labels \( k, c, m. \) Also shown is mass elongation \( x \) and the equilibrium position \( x = 0. \)

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Determine for \( y^{(4)} + y^{(2)} = x + 2e^x + 3 \sin x \) the corrected trial solution for \( y_p \) according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients!** The trial solution should be the one with fewest Euler solution atoms.

**Solution to Problem 7.**

The homogeneous equation \( y^{(4)} + y^{(2)} = 0 \) has solution \( y_h = c_1 + c_2 x + c_3 \cos x + c_4 \sin x \), because the characteristic polynomial has roots 0, 0, \( i \), \( -i \).

1. **Rule I** constructs an initial trial solution \( y \) from the list of independent Euler atoms

\[
e^x, \quad 1, \quad x, \quad \cos x, \quad \sin x.
\]

Linear combinations of these atoms are supposed to reproduce, by assignment of constants, all derivatives of \( F(x) = x + 2e^x + 3 \sin x \), which is the right side of the differential equation. Each of \( y_1 \) to \( y_4 \) in the display below is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of \( x \). For example, \( x = xe^{0x} \) strips to base atom \( e^{0x} \) or 1.

\[
\begin{align*}
y & = y_1 + y_2 + y_3 + y_4, \\
y_1 & = d_1 e^x, \\
y_2 & = d_2 + d_3 x, \\
y_3 & = d_4 \cos x, \\
y_4 & = d_5 \sin x.
\end{align*}
\]

2. **Rule II** is applied individually to each of \( y_1, y_2, y_3, y_4 \) to give the **corrected trial solution**

\[
\begin{align*}
y & = y_1 + y_2 + y_3 + y_4, \\
y_1 & = d_1 e^x, \\
y_2 & = x^2(d_2 + d_3 x), \\
y_3 & = x(d_4 \cos x), \\
y_4 & = x(d_5 \sin x).
\end{align*}
\]

The powers of \( x \) multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. The factor used is exactly \( x^s \) of the Edwards-Penney table, where \( s \) is the multiplicity of the characteristic equation root \( r \) that produced the related atom in the homogeneous solution \( y_h \). The atom in \( y_1 \) is not a solution of the homogeneous equation, therefore \( y_1 \) is unaltered.