Differential Equations 2280
Final Exam
Wednesday, 6 May 2015, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

(a) [40%] Given $y' = \dfrac{2x^2+x}{1+x}\left(\dfrac{y^4-2y^2+1}{y}\right)$,
find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly.

(b) [40%] Solve $2v'(t) = 5 + \dfrac{1}{t+1}v(t)$, $v(0) = 5$. Show all integrating factor steps.

(c) [10%] Solve the linear homogeneous equation $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$.

(d) [20%] The problem $\pi\sqrt{x+1}\,y' = y - 5\pi$ can be solved using superposition $y = y_h + y_p$. Find y_h and y_p.

Chapter 3: Linear Equations of Higher Order

(a) [10%] Solve for the general solution: $y'' + 4y' + 5y = 0$

(b) [20%] Solve for the general solution: $y^{(6)} + 16y^{(4)} = 0$

(c) [20%] Solve for the general solution, given the characteristic equation is $r(r+1)(r^3 - r)^2(r^2 + 2r + 5)^2 = 0$.

(d) [20%] Given $6x''(t) + 2x'(t) + 2x(t) = 5\cos(\omega t)$, which represents a damped forced spring-mass system with $m = 6$, $c = 2$, $k = 2$, answer the following questions.

True ☐ or False ☐. Practical mechanical resonance occurs for input frequency $\omega = \sqrt{5/2}$.

True ☐ or False ☐. The homogeneous problem is over-damped.

(e) [30%] Determine for $y^{(5)} + 4y^{(3)} = x + x^2 + e^x + x\cos(2x)$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Chapters 4 and 5: Systems of Differential Equations

(a) [20%] Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$.
Display eigenanalysis details for A.

(b) [10%] Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$.
Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$, using the eigenpairs from part (a).

(c) [30%] Find the general solution of the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

according to the Cayley-Hamilton-Ziebur Method, using the textbook's shortcut in chapter 4.

(d) [10%] Assume a 2×2 system $\frac{d}{dt}\vec{u} = A\vec{u}$ has a scalar general solution

$$x(t) = c_1 e^{-t} + c_2 e^{4t}, \quad y(t) = 3c_2 e^{-t} + (c_1 - c_2)e^{4t}.$$

Compute a fundamental matrix $\Phi(t)$.

(e) [10%] Assume given a 2×2 fundamental matrix. How do you find the exponential matrix from the fundamental matrix?

(f) [20%] Consider the scalar system

$$\begin{cases} x' &= 3x \\ y' &= x, \\ z' &= x + y \end{cases}$$

Solve the system by the most efficient method.

Chapter 6: Dynamical Systems

(a) [10%] Which of the four types *center, spiral, node, saddle* can be asymptotically stable at $t = \infty$? Explain your answer.

(b) [10%] The origin is an equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u}$. Is it an isolated equilibrium? Explain your answer.

In parts (c), (d), (e), consider the nonlinear dynamical system

$$x' = 14x - 2x^2 - xy, \quad y' = 16y - 2y^2 - xy. \qquad (1)$$

(c) [20%] Find the equilibrium points for the nonlinear system (1). There are four answers, one of which is $(4, 6)$.

(d) [30%] Consider again system (1). Classify the linearization at equilibrium point $(4, 6)$ as a node, spiral, center, saddle.

(e) [30%] Consider again system (1). What classification can be deduced for equilibrium $(4, 6)$ of this nonlinear system, according to the Pasting Theorem?

Chapter 7: Laplace Theory

(a) [20%] Explain all details in Laplace's Method, as applied to the differential equation

$$x''(t) + 2x'(t) = e^t, \quad x(0) = 0, \quad x'(0) = 0.$$

The solution $x(t) = \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}$ can be used as an answer check for your displayed details.

(b) [10%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s(s+2)^2}$.

(c) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(d) [30%] Solve by Laplace's Method the forced linear dynamical system

$$\begin{cases} x' &= x - y, \\ y' &= x + y + 2e^t, \end{cases}$$

subject to initial states $x(0) = 0$, $y(0) = 0$.

(e) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s-1}{s^2+2s+5}$.

(f) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}\left(t^2 e^{5t}\cos 8t\right)\right)\Big|_{s\to s+4}.$$

Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let $f_0(x) = x$ on the interval $-1 < x < 1$, $f_0(x) = 0$ for $x = \pm 1$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 2.

(a) [10%] Compute the Fourier cosine coefficients of $f(x)$.

(b) [10%] Find all values of x in $|x| < 4$ which will exhibit Gibb's over-shoot.

(c) [10%] State the theorem for term-by-term integration of Fourier series.

(d) [40%] **Heat Conduction in a Rod**. Solve the rod problem on $0 \le x \le L$, $t \ge 0$:

$$\begin{cases} u_t &= u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= 5\sin^2(2\pi x/L) \end{cases}$$

(e) [30%] **Vibration of a Finite String**. The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ on $0 < x < L$, $t > 0$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes (the superposition of the normal modes).

Solve the finite string vibration problem on $0 \le x \le 5$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= 100 u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(5,t) &= 0, \\ u(x,0) &= \sin(\pi x) + 5\sin(11\pi x), \\ u_t(x,0) &= 0 \end{cases}$$