Differential Equations 2280
Sample Final Exam
Wednesday, 6 May 2015, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2 \cot x + \frac{1250x^3}{1 + 25x^2} + x \ln(1 + x^2)$.

[The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

2. (Separable Equation Test)

The problem $y' = f(x, y)$ is said to be separable provided $f(x, y) = F(x)G(y)$ for some functions $F$ and $G$.

(a) [75%] Check (X) the problems that can be put into separable form, but don’t supply any details.

| $y' = -y(2xy + 1) + (2x + 3)y^2$ | $yy' = xy^2 + 5x^2y$ |
| $y' = e^{x+y} + e^y$ | $3y' + 5y = 10y^2$ |

(b) [25%] State a test which can verify that an equation is not separable. Apply the test to verify that $y' = x + \sqrt{|xy|}$ is not separable.

3. (Solve a Separable Equation)

Given $y^2y' = \frac{2x^2 + 3x}{1 + x^2} \left(\frac{125}{64} - y^3\right)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, do not solve for $y$ explicitly.

4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \frac{2}{3t + 1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x + 2} \frac{dy}{dx} = y$. The answer contains symbol $c$.

(c) [10%] The problem $2\sqrt{x + 2} y' = y - 5$ can be solved using the answer $y_h$ from part (b) plus superposition $y = y_h + y_p$. Find $y_p$. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.
5. (Stability)
   (a) [50%] Draw a phase line diagram for the differential equation
   \[
   \frac{dx}{dt} = 1000 \left(2 - \sqrt[3]{x}\right)^3 (2 + 3x)(9x^2 - 4)^8.
   \]
   Expected in the diagram are equilibrium points and signs of \(x'\) (or flow direction markers < and >).
   (b) [40%] Draw a phase portrait using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.
   (c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details. This is a paragraph of text, with only a few equations.

6. (ch3)
   (a) Solve for the general solutions:
   \[
   \begin{align*}
   (\text{a.1}) & \quad y'' + 4y' + 4y = 0, \\
   (\text{a.2}) & \quad y'' + 4y' = 0, \\
   (\text{a.3}) & \quad \text{Char. eq. } r(r-3)(r^3 - 9r)^2(r^2 + 4)^3 = 0.
   \end{align*}
   \]
   (b) Given \(6x''(t) + 7x'(t) + 2x(t) = 0\), which represents a damped spring-mass system with \(m = 6, \ c = 7, \ k = 2\), solve the differential equation [15\%] and classify the answer as over-damped, critically damped or under-damped [5\%]. Illustrate in a physical model drawing the meaning of constants \(m, \ c, \ k\) [5\%].

7. (ch3)
   Determine for \(y'' + y' = x + 2x^2 + x^3 + e^{-x} + x\sin x\) the shortest trial solution for \(y_p\) according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

8. (ch3)
   (a) [50\%] Find by undetermined coefficients the steady-state periodic solution for the equation \(x'' + 4x' + 6x = 10\cos(2t)\).
   (b) [50\%] Find by variation of parameters a particular solution \(y_p\) for the equation \(y'' + 3y' + 2y = xe^{2x}\).
   This instance has integration difficulties, therefore it is for a practise exam only. See Midterm 3 for an exam problem, required less integration effort.

9. (ch5)
The eigenanalysis method says that the system \(x' = Ax\) has general solution \(x(t) = c_1v_1e^{\lambda_1t} + c_2v_2e^{\lambda_2t} + c_3v_3e^{\lambda_3t}\). In the solution formula, \((\lambda_i, v_i), \ i = 1, 2, 3\) is an eigenpair of \(A\). Given
   \[
   A = \begin{bmatrix}
   5 & 1 & 1 \\
   1 & 5 & 1 \\
   0 & 0 & 7
   \end{bmatrix},
   \]
then
(a) [75%] Display eigenanalysis details for $A$.
(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

10. (ch5)

(a) [20%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) [40%] Putzer’s formula

$$e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I) + r_3(t)(A - \lambda_1 I)(A - \lambda_2 I)$$

uses the linear cascade

$$
\begin{align*}
    r_1' &= 2r_1, \quad r_1(0) = 1 \\
    r_2' &= 3r_2 + r_1, \quad r_2(0) = 0 \\
    r_3' &= 5r_3 + r_2, \quad r_3(0) = 0.
\end{align*}
$$

The general solution of $\mathbf{u}' = A\mathbf{u}$ according to Putzer’s spectral formula is $\mathbf{u} = e^{At}\mathbf{u}(0)$, where $e^{At}$ is defined above. Please compute all three coefficient functions $r_1, r_2, r_3$. **To save time**, don’t write out $e^{At}$ or the general solution.

(c) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur Method. In particular, display the equations that determine the three vectors in the general solution. **To save time**, don’t solve for the three vectors in the formula.

(d) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis Method. **To save time**, find one eigenpair explicitly, just to show how it is done, but don’t solve for the last two eigenpairs.

(e) [40%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to Laplace’s Method. **To save time**, use symbols for partial fraction constants and leave the symbols unevaluated.

11. (ch5) Do enough to make 100%  

(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$. Show details from either the eigenanalysis method or the Laplace method.

(b) [50%] Using the same matrix $A$ from part (a), display the solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(c) [50%] Using the same matrix $A$ from part (a), compute the exponential matrix $e^{At}$ by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$ where $\Phi(t)$ is any fundamental matrix, or via Putzer’s formula.
12. (ch5) Do both
   
   (a) [50%] Display the solution of $u' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} u$, $u(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies.
   
   (b) [50%] Display the variation of parameters formula for the system below. Then integrate to find $u_p(t)$ for $u' = Au$.

   $$u' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} u + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.$$  

13. (ch6)
   
   (a) Define asymptotically stable equilibrium for $u' = f(u)$, a 2-dimensional system.
   
   (b) Give examples of 2-dimensional systems of type saddle, spiral, center and node.
   
   (c) Give a 2-dimensional predator-prey example $u' = f(u)$ and explain the meaning of the variables in the model.

14. (ch6)

   Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify each linearization at an equilibrium as a node, spiral, center, saddle. What classifications can be deduced for the nonlinear system, according to the Paste Theorem?

   Some maple code for checking the answers:

   ```maple
   F:=unapply([14*x-x^2/2-y*x, 16*y-y^2/2-x*y],(x,y));
   Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
   Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
   Fx(0,0);Fy(0,0);Fx(28,0);Fy(28,0);Fx(0,32);Fy(0,32);Fx(0,32);Fy(0,32);
   ```

15. (ch6) Do enough to make 100%

   (a) [25%] Which of the four types center, spiral, node, saddle can be unstable at $t = \infty$? Explain your answer.
   
   (b) [25%] Give an example of a linear 2-dimensional system $u' = Au$ with a saddle at equilibrium point $x = y = 0$, and $A$ is not triangular.
   
   (c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.
   
   (d) [25%] Display a formula for the general solution of the equation $u' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} u$.

   Then explain why the system has a spiral at (0, 0).

   (e) [25%] Is the origin an isolated equilibrium point of the linear system $u' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u$? Explain your answer.

16. (ch7)

   (a) Define the direct Laplace Transform.
(b) Define Heaviside’s unit step function.
(c) Derive a Laplace integral formula for Heaviside’s unit step function.
(d) Explain all the steps in Laplace’s Method, as applied to the differential equation
\( x'(t) + 2x(t) = e^t, \ x(0) = 1. \)

17. (ch7)

(a) Solve \( \mathcal{L}(f(t)) = \frac{100}{(s^2 + 1)(s^2 + 4)} \) for \( f(t) \).
(b) Solve for \( f(t) \) in the equation \( \mathcal{L}(f(t)) = \frac{1}{s^2(s - 3)} \).
(c) Find \( \mathcal{L}(f) \) given \( f(t) = (-t)e^{2t} \sin(3t) \).
(d) Find \( \mathcal{L}(f) \) where \( f(t) \) is the periodic function of period 2 equal to \( t/2 \) on \( 0 \leq t \leq 2 \) (sawtooth wave).

18. (ch7)

(a) Solve \( y'' + 4y' + 4y = t^2, \ y(0) = y'(0) = 0 \) by Laplace’s Method.
(b) Solve \( x''' + x'' - 6x' = 0, \ x(0) = x'(0) = 0, \ x''(0) = 1 \) by Laplace’s Method.
(c) Solve the system \( x' = x + y, \ y' = x - y + e^t, \ x(0) = 0, \ y(0) = 0 \) by Laplace’s Method.

19. (ch7)

(a) [25%] Solve by Laplace’s method \( x'' + x = \cos t, \ x(0) = x'(0) = 0. \)
(b) [10%] Does there exist \( f(t) \) of exponential order such that \( \mathcal{L}(f(t)) = \frac{s}{s + 1} \)? Details required.
(c) [15%] Linearity \( \mathcal{L}(c_1f + c_2g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g) \) is one Laplace rule. State four other Laplace rules. Forward and backward table entries are not rules, which means \( \mathcal{L}(1) = 1/s \) doesn’t count.
(d) [25%] Solve by Laplace’s resolvent method
\[
\begin{align*}
x'(t) & = x(t) + y(t), \\
y'(t) & = 2x(t),
\end{align*}
\]
with initial conditions \( x(0) = -1, \ y(0) = 2. \)
(e) [25%] Derive \( y(t) = \int_0^t \sin(t - u)f(u)du \) by Laplace transform methods from the forced oscillator problem
\( y''(t) + y(t) = f(t), \ y(0) = y'(0) = 0. \)
20. (ch7)
(a) [25%] Solve \( L(f(t)) = \frac{10}{(s^2 + 8)(s^2 + 4)} \) for \( f(t) \).
(b) [25%] Solve for \( f(t) \) in the equation \( L(f(t)) = \frac{s + 1}{s^2(s + 2)} \).
(c) [20%] Solve for \( f(t) \) in the equation \( L(f(t)) = \frac{s - 1}{s^2 + 2s + 5} \).
(d) [10%] Solve for \( f(t) \) in the relation \( L(f) = \frac{d}{ds} L(t^2 \sin 3t) \).
(e) [10%] Solve for \( f(t) \) in the relation \( L(f) = \left. L\left(t^3 e^{9t} \cos 8t\right)\right|_{s \to s+3} \).

21. (ch9)
(a) Find the Fourier sine and cosine coefficients for the 2-periodic function \( f(t) \) equal to \( t/2 \) on \( 0 \leq t \leq 2 \).
(b) State Fourier’s convergence theorem.
(c) State the results for term-by-term integration and differentiation of Fourier series.

22. (ch9)
(a) Find a steady-state periodic solution by Fourier’s method for \( x'' + x = F(t) \), where \( F(t) \) is 2-periodic and equal to 10 on \( 0 < t < 1 \), equal to \(-10\) on \( 1 < t < 2 \).
(b) Display Fourier’s Model for the solution to the heat problem \( u_t = u_{xx} \), \( u(0,t) = u(1,t) = 0 \), \( u(x,0) = f(x) \) on \( 0 \leq x \leq 1 \), \( t \geq 0 \).
(c) Solve \( u_t = u_{xx} \), \( u(0,t) = u(\pi,t) = 0 \), \( u(x,0) = 80 \sin^3 x \) on \( 0 \leq x \leq \pi \), \( t \geq 0 \).

23. (Vibration of a Finite String)
The normal modes for the string equation \( u_{tt} = c^2 u_{xx} \) are given by the functions
\[
\sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi ct}{L} \right), \quad \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi ct}{L} \right).
\]
It is known that each normal mode is a solution of the string equation and that the problem below has solution \( u(x,t) \) equal to an infinite series of constants times normal modes.
Solve the finite string vibration problem on \( 0 \leq x \leq 2 \), \( t > 0 \),
\[
\begin{align*}
u_{tt} &= c^2 u_{xx}, \\
u(0,t) &= 0, \\
u(2,t) &= 0, \\
u(x,0) &= 0, \\
u_t(x,0) &= -11 \sin(5\pi x).
\end{align*}
\]
24. (Periodic Functions)

(a) [30\%] Find the period of \( f(x) = \sin(x) \cos(2x) + \sin(2x) \cos(x) \).

(b) [40\%] Let \( p = 5 \). If \( f(x) \) is the odd \( 2p \)-periodic extension to \((-\infty, \infty)\) of the function \( f_0(x) = 100xe^{10x} \) on \( 0 \leq x \leq p \), then find \( f(11.3) \). The answer is not to be simplified or evaluated to a decimal.

(c) [30\%] Mark the expressions which are periodic with letter P, those odd with O and those even with E.

\[
\sin(\cos(2x)) \quad \ln|2 + \sin(x)| \quad \sin(2x) \cos(x) \quad \frac{1 + \sin(x)}{2 + \cos(x)}
\]

25. (Fourier Series)

Let \( f_0(x) = x \) on the interval \( 0 < x < 2 \), \( f_0(x) = -x \) on \(-2 < x < 0 \), \( f_0(x) = 0 \) for \( x = 0 \), \( f_0(x) = 2 \) at \( x = \pm2 \). Let \( f(x) \) be the periodic extension of \( f_0 \) to the whole real line, of period 4.

(a) [80\%] Compute the Fourier coefficients of \( f(x) \) (defined above) for the terms \( \sin(67\pi x) \) and \( \cos(2\pi x) \). Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20\%] Which values of \( x \) in \( |x| < 12 \) might exhibit Gibb’s over-shoot?

26. (Cosine and Sine Series)

Find the first nonzero term in the sine series expansion of \( f(x) \), formed as the odd \( 2\pi \)-periodic extension of the function \( \sin(x) \cos(x) \) on \( 0 < x < \pi \). Leave the Fourier coefficient in integral form, unevaluated, unless you can compute the value in a minute or two.

27. (Convergence of Fourier Series)

(a) [30\%] Dirichlet’s kernel formula can be used to evaluate the sum \( \cos(2x) + \cos(4x) + \cos(6x) + \cos(8x) \). Report its value according to that formula. [Not on the final exam 2015]

(b) [40\%] The Fourier Convergence Theorem for piecewise smooth functions applies to continuously differentiable functions of period \( p \). Re-state the Fourier Convergence Theorem for the special case of a \( p \)-periodic continuously differentiable function. It is necessary to translate the results for interval \(-\pi \leq x \leq \pi \) to the interval \(-p \leq x \leq p \) and simplify the value to which the Fourier series converges.

(c) [30\%] Give an example of a function \( f(x) \) periodic of period 2 that has a Gibb’s over-shoot at the integers \( x = 0, \pm2, \pm4, \ldots \), (all \( \pm2n \)) and nowhere else.