Differential Equations 2280
Sample Midterm Exam 3 with Solutions
Exam Date: 24 April 2015 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exams 1, 2.

Chapter 3

1. (Linear Constant Equations of Order \(n\))
   (a) Find by variation of parameters a particular solution \(y_p\) for the equation \(y'' = 1 - x\). Show all steps in variation of parameters. Check the answer by quadrature.
   (b) A particular solution of the equation \(mx'' + cx' + kx = F_0 \cos(2t)\) happens to be \(x(t) = 11 \cos 2t + e^{-t} \sin \sqrt{11}t - \sqrt{11} \sin 2t\). Assume \(m, c, k\) all positive. Find the unique periodic steady-state solution \(x_{ss}\).
   (c) A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions \(2e^{3x} + 4x\) and \(xe^{3x}\). Write a formula for the general solution.
   (d) Find the Beats solution for the forced undamped spring-mass problem
   \[
x'' + 64x = 40 \cos(4t), \quad x(0) = x'(0) = 0.
   \]
   It is known that this solution is the sum of two harmonic oscillations of different frequencies. To save time, don’t convert to phase-amplitude form.
   (e) Write the solution \(x(t)\) of
   \[
x''(t) + 25x(t) = 180 \sin(4t), \quad x(0) = x'(0) = 0,
   \]
as the sum of two harmonic oscillations of different natural frequencies.
   To save time, don’t convert to phase-amplitude form.
   (f) Find the steady-state periodic solution for the forced spring-mass system \(x'' + 2x' + 2x = 5 \sin(t)\).
   (g) Given \(5x''(t) + 2x'(t) + 4x(t) = 0\), which represents a damped spring-mass system with \(m = 5, c = 2, k = 4\), determine if the equation is over-damped, critically damped or under-damped.
   To save time, do not solve for \(x(t)\)!
   (h) Determine the practical resonance frequency \(\omega\) for the electric current equation
   \[
   2I'' + 7I' + 50I = 100\omega \cos(\omega t).
   \]
   (i) Given the forced spring-mass system \(x'' + 2x' + 17x = 82 \sin(5t)\), find the steady-state periodic solution.
   (j) Let \(f(x) = x^3e^{1.2x} + x^2e^{-x} \sin(x)\). Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has \(f(x)\) as a solution. To save time, do not expand the polynomial and do not find the differential equation.
Answers and Solution Details:

Part (a) Answer: \( y_p = \frac{x^2}{2} - \frac{x^3}{6} \).

Variation of Parameters.
Solve \( y'' = 0 \) to get \( y_h = c_1y_1 + c_2y_2, \ y_1 = 1, \ y_2 = x \). Compute the Wronskian \( W = y_1y_2' - y_1'y_2 = 1 \). Then for \( f(t) = 1 - x \),

\[
y_p = y_1 \left( \int y_2 \frac{-f}{W} \, dx + y_2 \int \frac{f}{W} \, dx \right),
\]

\[
y_p = 1 \left( \int -x(1-x) \, dx + x \int 1(1-x) \, dx \right),
\]

\[
y_p = \int -x^2/2 + \frac{x^3}{3} + x(x - x^2/2),
\]

\[
y_p = \frac{x^2}{2} - \frac{x^3}{6}.
\]

This answer is checked by quadrature, applied twice to \( y'' = 1 - x \) with initial conditions zero.

Part (b) It has to be the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then \( x_{ss}(t) = 11 \cos 2t - \sqrt{11} \sin 2t \).

Part (c) In order for \( xe^{3x} \) to be a solution, the general solution must have Euler atoms \( e^{3x}, xe^{3x} \). Then the first solution \( 2e^{3x} + 4x \) minus 2 times the Euler atom \( e^{3x} \) must be a solution, therefore \( x \) is a solution, resulting in Euler atoms \( 1, x \). The general solution is then a linear combination of the four Euler atoms:

\[
y = c_1(1) + c_2(x) + c_3(e^{3x}) + c_4(xe^{3x}).
\]

Part (d) Use undetermined coefficients trial solution \( x = d_1 \cos 4t + d_2 \sin 4t \). Then \( d_1 = 5/6, \ d_2 = 0 \), and finally \( x_p(t) = (5/6) \cos(4t) \). The characteristic equation \( r^2 + 64 = 0 \) has roots \( \pm 8i \) with corresponding Euler solution \( \cos(8t), \sin(8t) \). Then \( x_h(t) = c_1 \cos(8t) + c_2 \sin(8t) \). The general solution is \( x = x_h + x_p \). Now use \( x(0) = x'(0) = 0 \) to determine \( c_1 = -5/6, c_2 = 0 \), which implies the particular solution \( x(t) = -\frac{5}{6} \cos(8t) + \frac{2}{5} \cos(4t) \).

Part (e) The answer is \( x(t) = -16 \sin(5t) + 20 \sin(4t) \) by the method of undetermined coefficients.

Rule I: \( x = d_1 \cos(4t) + d_2 \sin(4t) \). Rule II does not apply due to natural frequency \( \sqrt{25} = 5 \) not equal to the frequency of the trial solution (no conflict). Substitute the trial solution into \( x''(t) + 25x(t) = 180 \sin(4t) \) to get \( 9d_1 \cos(4t) + 9d_2 \sin(4t) = 180 \sin(4t) \). Match coefficients, to arrive at the equations \( 9d_1 = 0, 9d_2 = 0 \). Then \( d_1 = 0, d_2 = 20 \) and \( x_p(t) = 20 \sin(4t) \). Lastly, add the homogeneous solution to obtain \( x(t) = x_h + x_p = c_1 \cos(5t) + c_2 \sin(5t) + 20 \sin(4t) \). Use the initial condition relations \( x(0) = 0, x'(0) = 0 \) to obtain the equations \( \cos(0)c_1 + \sin(0)c_2 + 20 \sin(0) = 0, -5 \sin(0)c_1 + 5 \cos(0)c_2 + 80 \cos(0) = 0 \). Solve for the coefficients \( c_1 = 0, c_2 = -16 \).

Part (f) The answer is \( x = \sin t - 2 \cos t \) by the method of undetermined coefficients.

Rule I: the trial solution \( x(t) \) is a linear combination of the Euler atoms found in \( f(x) = 5 \sin t \). Then \( x(t) = d_1 \cos(t) + d_2 \sin(t) \). Rule II does not apply, because solutions of the homogeneous problem contain negative exponential factors (no conflict). Substitute the trial solution into \( x'' + 2x' + 2x = 5 \sin(t) \) to get \( -2d_1 + d_2 \sin(t) + (d_1 + 2d_2) \cos(t) = 5 \sin(t) \). Match coefficients to find the system of equations \( -2d_1 + d_2 = 5, \ (d_1 + 2d_2) = 0 \). Solve for the coefficients \( d_1 = -2, d_2 = 1 \).

Part (g) Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is \( b^2 - 4ac = 2^2 - 4(5)(4) = 19(-4) \), therefore there are two complex conjugate roots and the equation is under-damped. Alternatively, factor \( 5r^2 + 2r + 4 \) to obtain roots \( -1 \pm \sqrt{19i}/5 \) and then classify as under-damped.

Part (h) The resonant frequency is \( \omega = 1/\sqrt{L/C} = 1/\sqrt{2/50} = \sqrt{25} = 5 \). The solution uses the theory in the textbook, section 3.7, which says that electrical resonance occurs for \( \omega = 1/\sqrt{L/C} \).
Part (i) The answer is \( x(t) = -5 \cos(5t) - 4 \sin(5t) \) by undetermined coefficients.

Rule I: The trial solution is \( x_p(t) = A \cos(5t) + B \sin(5t) \). Rule II: because the homogeneous solution \( x_h(t) \) has limit zero at \( t = \infty \), then Rule II does not apply (no conflict). Substitute the trial solution into the differential equation. Then \(-8A \cos(5t) - 8B \sin(5t) - 10A \sin(5t) + 10B \cos(5t) = 82 \sin(5t)\). Matching coefficients of sine and cosine gives the equations \(-8A + 10B = 0, -10A - 8B = 82\). Solving, \( A = -5, B = -4 \). Then \( x_p(t) = -5 \cos(5t) - 4 \sin(5t) \) is the unique periodic steady-state solution.

Part (j) The characteristic polynomial is the expansion \((r - 1.2)^4((r + 1)^2 + 1)^3\). Because \( xe^{ax} \) is an Euler solution atom for the differential equation if and only if \( e^{ax}, xe^{ax}, x^2 e^{ax}, x^3 e^{ax} \) are Euler solution atoms, then the characteristic equation must have roots \( 1.2, 1.2, 1.2, 1.2 \), listing according to multiplicity. Similarly, \( x^2 e^{-x} \sin(x) \) is an Euler solution atom for the differential equation if and only if \( -1 \pm i, -1 \pm i, -1 \pm i \) are roots of the characteristic equation. There is a total of 10 roots with product of the factors \((r - 1)^4((r + 1)^2 + 1)^3\) equal to the 10th degree characteristic polynomial.
Chapters 4 and 5

2. (Systems of Differential Equations)

Background. Let $A$ be a real $3 \times 3$ matrix with eigenpairs $(\lambda_1, v_1)$, $(\lambda_2, v_2)$, $(\lambda_3, v_3)$. The eigenanalysis method says that the $3 \times 3$ system $\mathbf{x}' = A\mathbf{x}$ has general solution

$$\mathbf{x}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}.$$

Background. Let $A$ be an $n \times n$ real matrix. The method called Cayley-Hamilton-Ziebur is based upon the result

The components of solution $\mathbf{x}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$.

Background. Let $A$ be an $n \times n$ real matrix. An augmented matrix $\Phi(t)$ of $n$ independent solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is called a fundamental matrix. It is known that the general solution is $\mathbf{x}(t) = \Phi(t)c$, where $c$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

(a) Display eigenanalysis details for the $3 \times 3$ matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) The $3 \times 3$ triangular matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models. Using the linear integrating factor method, starting with component $x_3(t)$, find the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) The exponential matrix $e^{At}$ is defined to be a fundamental matrix $\Psi(t)$ selected such that $\Psi(0) = I$, the $n \times n$ identity matrix. Justify the formula $e^{At} = \Phi(t)\Phi(0)^{-1}$, valid for any fundamental matrix $\Phi(t)$.

(d) Let $A$ denote a $2 \times 2$ matrix. Assume $\mathbf{x}'(t) = A\mathbf{x}(t)$ has scalar general solution $x_1 = c_1 e^t + c_2 e^{2t}$,

$$x_2 = (c_1 - c_2)e^t + 2c_1 + c_2)e^{2t},$$

where $c_1, c_2$ are arbitrary constants. Find a fundamental matrix $\Phi(t)$ and then go on to find $e^{At}$ from the formula in part (c) above.

(e) Let $A$ denote a $2 \times 2$ matrix and consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Assume fundamental matrix

$$\Phi(t) = \begin{pmatrix} e^t & e^{2t} \\ 2e^t & -e^{2t} \end{pmatrix}.$$ 

Find the $2 \times 2$ matrix $A$.

(f) The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 3x + y, \quad y' = -x + 3y,$$

which has complex eigenvalues $\lambda = 3 \pm i$. Show the details of the method, then go on to report a fundamental matrix $\Phi(t)$.

Remark. The vector general solution is $\mathbf{x}(t) = \Phi(t)c$, which contains no complex numbers. Reference: 4.1, Examples 6,7,8.
Answers and Solution Details:

Part (a) The details should solve the equation \(|A - \lambda I| = 0\) for the three eigenvalues \(\lambda = 5, 4, 3\). Then solve the three systems \((A - \lambda I)\vec{v} = 0\) for eigenvector \(\vec{v}\), for \(\lambda = 5, 4, 3\). The eigenpairs are

\[
\begin{pmatrix} 5, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix}; \begin{pmatrix} 4, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ 1 \end{pmatrix}; \begin{pmatrix} 3, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ 0 \end{pmatrix}.
\]

The eigenanalysis method implies

\[
x(t) = c_1e^{5t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2e^{4t}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.
\]

Part (b) Write the system in scalar form

\[
x' = 3x + y + z,
y' = 4y + z,
z' = 5z.
\]

Solve the last equation as

\[
z = \text{constant}\frac{\text{integrating factor}}{c_3e^{5t}} = c_3e^{5t}.
\]

The second equation is

\[
y' = 4y + c_3e^{5t}
\]

The linear integrating factor method applies.

\[
y' - 4y = c_3e^{-5t}
\]

\[
(Wy)' = c_3e^{5t}, \text{ where } W = e^{-4t},
\]

\[
(Wy)' = c_3We^{5t}
\]

\[
(e^{-4t}y)' = c_3e^{-4t}e^{5t}
\]

\[
e^{-4t}y = c_3e^{t} + c_2.
\]

Stuf these two expressions into the first differential equation:

\[
x' = 3x + y + z = 3x + 2c_3e^{5t} + c_2e^{4t}
\]

Then solve with the linear integrating factor method.

\[
x' - 3x = 2c_3e^{5t} + c_2e^{4t}
\]

\[
(Wx)' = 2c_3e^{5t} + c_2e^{4t}, \text{ where } W = e^{-3t}. \text{ Cross-multiply}:
\]

\[
(e^{-3t}x)' = 2c_3e^{5t}e^{-3t} + c_2e^{4t}e^{-3t}, \text{ then integrate}:
\]

\[
e^{-3t}x = c_3e^{2t} + c_2e^{t} + c_1
\]

\[
e^{-3t}x = c_3e^{2t} + c_2e^{t} + c_1, \text{ divide by } e^{-3t}:
\]

\[
x = c_3e^{5t} + c_2e^{4t} + c_1e^{3t}
\]

Part (c) The question reduces to showing that \(e^{At}\) and \(\Phi(t)\Phi(0)^{-1}\) have equal columns. This is done by showing that the matching columns are solutions of \(\vec{u}' = A\vec{u}\) with the same initial condition \(\vec{u}(0)\), then apply Picard’s theorem on uniqueness of initial value problems.

Part (d) Take partial derivatives on the symbols \(c_1, c_2\) to find vector solutions \(\vec{v}_1(t), \vec{v}_2(t)\). Define \(\Phi(t)\) to be the augmented matrix of \(\vec{v}_1(t), \vec{v}_2(t)\). Compute \(\Phi(0)^{-1}\), then multiply on the right of \(\Phi(t)\) to obtain...
\[ e^{At} = \Phi(t)\Phi(0)^{-1}. \] Check the answer in a computer algebra system or using Putzer’s formula.

**Part (e)** The equation \( \Phi'(t) = A\Phi(t) \) holds for every \( t \). Choose \( t = 0 \) and then solve for \( A = \Phi'(0)\Phi(0)^{-1} \).

**Part (f)** By C-H-Z, \( x = c_1 e^{3t} \cos(t) + c_2 e^{3t} \sin(t) \). Isolate \( y \) from the first differential equation \( x' = 3x + y \), obtaining the formula \( y = x' - 3x = -c_1 e^{3t} \sin(t) + c_2 e^{3t} \cos(t) \). A fundamental matrix is found by taking partial derivatives on the symbols \( c_1, c_2 \). The answer is exactly the Jacobian matrix of \( \begin{pmatrix} x \\ y \end{pmatrix} \) with respect to variables \( c_1, c_2 \).

\[ \Phi(t) = \begin{pmatrix} e^{3t} \cos(t) & e^{3t} \sin(t) \\ -e^{3t} \sin(t) & e^{3t} \cos(t) \end{pmatrix}. \]
3. (Linear and Nonlinear Dynamical Systems)
   (a) Determine whether the unique equilibrium \( \vec{u} = \vec{0} \) is stable or unstable. Then classify the equilibrium point \( \vec{u} = \vec{0} \) as a saddle, center, spiral or node.
   \[
   \vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}
   \]

   (b) Determine whether the unique equilibrium \( \vec{u} = \vec{0} \) is stable or unstable. Then classify the equilibrium point \( \vec{u} = \vec{0} \) as a saddle, center, spiral or node.
   \[
   \vec{u}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{u}
   \]

   (c) Consider the nonlinear dynamical system
   \[
   \begin{align*}
   x' &= x - 2y^2 - y + 32, \\
   y' &= 2x^2 - 2xy.
   \end{align*}
   \]
   An equilibrium point is \( x = 4, y = 4 \). Compute the Jacobian matrix \( A = J(4, 4) \) of the linearized system at this equilibrium point.

   (d) Consider the nonlinear dynamical system
   \[
   \begin{align*}
   x' &= -x - 2y^2 - y + 32, \\
   y' &= 2x^2 + 2xy.
   \end{align*}
   \]
   An equilibrium point is \( x = -4, y = 4 \). Compute the Jacobian matrix \( A = J(-4, 4) \) of the linearized system at this equilibrium point.

   (e) Consider the nonlinear dynamical system
   \[
   \begin{align*}
   x' &= -4x + 4y + 9 - x^2, \\
   y' &= 3x - 3y.
   \end{align*}
   \]
   At equilibrium point \( x = 3, y = 3 \), the Jacobian matrix is \( A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix} \).

      1. Determine the stability at \( t = \infty \) and the phase portrait classification saddle, center, spiral or node at \( \vec{u} = \vec{0} \) for the linear system \( \frac{d}{dt} \vec{u} = A\vec{u} \).

      2. Apply the Pasting Theorem to classify \( x = 3, y = 3 \) as a saddle, center, spiral or node for the nonlinear dynamical system. Discuss all details of the application of the theorem. Details count 75%.

   (f) Consider the nonlinear dynamical system
   \[
   \begin{align*}
   x' &= -4x - 4y + 9 - x^2, \\
   y' &= 3x + 3y.
   \end{align*}
   \]
   At equilibrium point \( x = 3, y = -3 \), the Jacobian matrix is \( A = J(3, -3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix} \).

      Linearization. Determine the stability at \( t = \infty \) and the phase portrait classification saddle, center, spiral or node at \( \vec{u} = \vec{0} \) for the linear dynamical system \( \frac{d}{dt} \vec{u} = A\vec{u} \).

      Nonlinear System. Apply the Pasting Theorem to classify \( x = 3, y = -3 \) as a saddle, center, spiral or node for the nonlinear dynamical system. Discuss all details of the application of the theorem. Details count 75%.
Answers and Solution Details:

Part (a) It is an unstable spiral. Details: The eigenvalues of $A$ are roots of $r^2 - 2r + 5 = (r - 1)^2 + 4 = 0$, which are complex conjugate roots $1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^t \cos 2t$, $e^t \sin 2t$ have limit zero at $t = -\infty$, therefore the system is stable at $t = -\infty$ and unstable at $t = \infty$. So it must be a spiral [centers have no exponentials]. Report: unstable spiral.

Part (b) It is a stable spiral. Details: The eigenvalues of $A$ are roots of $r^2 + 2r + 5 = (r + 1)^2 + 4 = 0$, which are complex conjugate roots $-1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^{-t} \cos 2t$, $e^{-t} \sin 2t$ have limit zero at $t = \infty$, therefore the system is stable at $t = \infty$ and unstable at $t = -\infty$. So it must be a spiral [centers have no exponentials]. Report: stable spiral.

Part (c) The Jacobian is $J(x, y) = \begin{pmatrix} 1 & -4y - 1 \\ 4x - 2y & -2x \end{pmatrix}$. Then $A = J(4, 4) = \begin{pmatrix} 1 & -17 \\ 8 & -8 \end{pmatrix}$.

Part (d) The Jacobian is $J(x, y) = \begin{pmatrix} -1 & -4y - 1 \\ 4x + 2y & 2x \end{pmatrix}$. Then $A = J(-4, 4) = \begin{pmatrix} -1 & -17 \\ -8 & -8 \end{pmatrix}$.

Part (e) (1) The Jacobian is $J(x, y) = \begin{pmatrix} -4 - 2x & 4 \\ 3 & -3 \end{pmatrix}$. Then $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$. The eigenvalues of $A$ are found from $r^2 + 13r + 18 = 0$, giving distinct real negative roots $-\frac{13}{2} \pm \frac{1}{2} \sqrt{97}$. Because there are no trig functions in the Euler solution atoms, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms limit to zero at $t = \infty$, therefore it is a node and we report a stable node for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$.

(2) Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: stable node at $x = 3$, $y = 3$.

Part (f)

Linearization. The Jacobian is $J(x, y) = \begin{pmatrix} -4 - 2x & -4 \\ 3 & 3 \end{pmatrix}$. Then $A = J(3, 3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$. The eigenvalues of $A$ are found from $r^2 + 7r - 18 = 0$, giving distinct real roots $2, -9$. Because there are no trig functions in the Euler solution atoms $e^{2t}, e^{-9t}$, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms do not limit to zero at $t = \infty$ or $t = -\infty$, therefore it is a saddle and we report a unstable saddle for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$.

Nonlinear System. Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: unstable saddle at $x = 3$, $y = 3$.
Final Exam Problems

**Chapter 5.** Solve a homogeneous system $u' = Au$, $u(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ using the matrix exponential, Zeibur's method, Laplace resolvent and eigenanalysis.

**Chapter 5.** Solve a non-homogeneous system $u' = Au + F(t)$, $u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $F(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ using variation of parameters.