Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exam 1.

1. (Chapter 3)  
(a) [50%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = -10 \sin(t)$.
(b) [50%] Find by variation of parameters a particular solution $y_p$ for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature.
2. (Chapters 1, 2, 3)

(2a) [20%] Solve $2v'(t) = -8 + \frac{2}{2t + 1}v(t)$, $v(0) = -4$. Show all integrating factor steps.

(2b) [10%] Solve for the general solution: $y'' + 4y' + 6y = 0$.

(2c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is $r(r^2 + r)^2(r^2 + 9)^2 = 0$.

(2d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution $y = x + \sin \sqrt{2}x + e^{-x} \cos 3x$.

(2e) [15%] A particular solution of the equation $mx'' + cx' + kx = F_0 \cos(2t)$ happens to be $x(t) = 11 \cos 2t + e^{-t} \sin \sqrt{11}t - \sqrt{11} \sin 2t$. Assume $m, c, k$ all positive. Find the unique periodic steady-state solution $x_{ss}$.

(2f) [25%] Determine for $y''' + y'' = 100x^2 + \sin x$ the shortest trial solution for $y_p$ according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Use this page to start your solution. Attach extra pages as needed, then staple.
3. (Laplace Theory)
   (a) [50%] Solve by Laplace’s method $x'' + 2x' + x = e^t$, $x(0) = x'(0) = 0$.
   (b) [25%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation
   \[ \frac{d}{ds} \mathcal{L}(f(t)) \bigg|_{s \to (s-3)} = \mathcal{L}(f(t) - t). \]
   (c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods, explicitly using the Convolu-
   tion Theorem, for the problem
   \[ y''(t) + 4y'(t) + 4y(t) = f(t), \quad y(0) = y'(0) = 0. \]
4. (Laplace Theory)

(4a) [20%] Explain Laplace’s Method, as applied to the differential equation \(x'(t) + 2x(t) = e^t, \ x(0) = 1\).

(4b) [15%] Solve \(\mathcal{L}(f(t)) = \frac{100}{(s^2 + 4)(s^2 + 9)}\) for \(f(t)\).

(4c) [15%] Solve for \(f(t)\) in the equation \(\mathcal{L}(f(t)) = \frac{1}{s^2(s + 3)}\).

(4d) [10%] Find \(\mathcal{L}(f)\) given \(f(t) = (-t)e^{2t} \sin(3t)\).

(4e) [20%] Solve \(x'''' + x''' = 0, \ x(0) = 1, \ x'(0) = 0, \ x''(0) = 0\) by Laplace’s Method.

(4f) [20%] Solve the system \(x' = x + y, \ y' = x - y + 2, \ x(0) = 0, \ y(0) = 0\) by Laplace’s Method.

Use this page to start your solution. Attach extra pages as needed, then staple.
5. (Laplace Theory)
(a) [30%] Solve \( \mathcal{L}(f(t)) = \frac{1}{(s^2 + s)(s^2 - s)} \) for \( f(t) \).

(b) [20%] Solve for \( f(t) \) in the equation \( \mathcal{L}(f(t)) = \frac{s + 1}{s^2 + 4s + 5} \).

(c) [20%] Let \( u(t) \) denote the unit step. Solve for \( f(t) \) in the relation

\[
\mathcal{L}(f(t)) = \frac{d}{ds} \mathcal{L}(u(t - 1) \sin 2t)
\]

(d) [30%] Compute \( \mathcal{L}(e^{2t}f(t)) \) for

\[
f(t) = \frac{e^t - e^{-t}}{t}.
\]
6. (Systems of Differential Equations)

The eigenanalysis method says that, for a $3 \times 3$ system $x' = Ax$, the general solution is $x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, v_i), i = 1, 2, 3$, is an eigenpair of $A$. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then

(a) [75%] Display eigenanalysis details for $A$.

(b) [25%] Display the solution $x(t)$ of $x'(t) = Ax(t)$. (c) Repeat (a), (b) for the matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix}$.

Use this page to start your solution. Attach extra pages as needed, then staple.
7. (Systems of Differential Equations)

(a) [40%] Find the eigenvalues of the matrix

\[
A = \begin{bmatrix}
4 & 1 & -1 & 0 \\
1 & 4 & -2 & 1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 2 & 4 \\
\end{bmatrix}.
\]

(b) [60%] Display the general solution of \( u' = Au \) according to Putzer’s spectral formula. Leave matrix products unexpanded, in order to save time. However, do compute the coefficient functions \( r_1, r_2, r_3, r_4 \). The correct answer for \( r_4 \), using \( \lambda \) in increasing magnitude, is

\[
y(x) = \frac{1}{6}e^{5t} - \frac{1}{2}e^{4t} + \frac{1}{2}e^{3t} - \frac{1}{6}e^{2t}.
\]
8. (Systems of Differential Equations)

(a) [30%] The eigenvalues are 3, 5 for the matrix
A = \[
\begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}
\].

Display the general solution of \(u' = Au\) according to Putzer’s spectral formula. Don’t expand matrix products, in order to save time.

(b) [20%] Using the same matrix \(A\) from part (a), display the solution of \(u' = Au\) according to the Cayley-Hamilton-Ziebur Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors. Assume initial condition \(\vec{u}_0 = \left(\begin{array}{c} 1 \\ -1 \end{array}\right)\).

(c) [30%] Using the same matrix \(A\) from part (a), compute explicitly all four entries of the exponential matrix \(e^{At}\) by any known method. Use either Putzer’s formula or the formula \(e^{At} = \Phi(t)\Phi^{-1}(0)\), where \(\Phi\) is a fundamental matrix.

(e) [20%] Display the solution of \(u' = Au\), \(\vec{u}(0) = \left(\begin{array}{c} 1 \\ -1 \end{array}\right)\).