11.7 Nonhomogeneous Linear Systems

Variation of Parameters

The **method of variation of parameters** is a general method for solving a linear nonhomogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t).$$

Historically, it was a trial solution method, whereby the nonhomogeneous system is solved using a trial solution of the form

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0(t).$$

In this formula, $\mathbf{x}_0(t)$ is a vector function to be determined. The method is imagined to originate by varying $\mathbf{x}_0$ in the general solution $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0$ of the linear homogenous system $\mathbf{x}' = A\mathbf{x}$. Hence was coined the names *variation of parameters* and *variation of constants.*

Modern use of variation of parameters is through a formula, memorized for routine use.

Theorem 28 (Variation of Parameters for Systems)
Let A be a constant $n \times n$ matrix and $\mathbf{F}(t)$ a continuous function near $t = t_0$. The unique solution $\mathbf{x}(t)$ of the matrix initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{F}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

is given by the **variation of parameters formula**

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_{t_0}^{t} e^{-rA}\mathbf{F}(r)dr. \tag{1}$$

Proof of (1). Define

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^{t} e^{-rA}\mathbf{F}(r)dr.$$

To show (1) holds, we must verify $\mathbf{x}(t) = e^{At}\mathbf{u}(t)$. First, the function $\mathbf{u}(t)$ is differentiable with continuous derivative $e^{-tA}\mathbf{F}(t)$, by the fundamental theorem of calculus applied to each of its components. The product rule of calculus applies to give

$$\begin{aligned}
\mathbf{x}'(t) &= \left(e^{At}\right)'\mathbf{u}(t) + e^{At}\mathbf{u}'(t) \\
&= Ae^{At}\mathbf{u}(t) + e^{At}e^{-At}\mathbf{F}(t) \\
&= A\mathbf{x}(t) + \mathbf{F}(t).
\end{aligned}$$

Therefore, $\mathbf{x}(t)$ satisfies the differential equation $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$. Because $\mathbf{u}(t_0) = \mathbf{x}_0$, then $\mathbf{x}(t_0) = \mathbf{x}_0$, which shows the initial condition is also satisfied. The proof is complete.

Undetermined Coefficients

The trial solution method known as the method of undetermined coefficients can be applied to vector-matrix systems $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$ when the components of $\mathbf{F}$ are sums of terms of the form

$$(\text{polynomial in } t)e^{at}(\cos(bt) \text{ or } \sin(bt)).$$

Such terms are known as **atoms**. It is usually efficient to write $\mathbf{F}$ in terms of the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix I, as the combination

$$\mathbf{F}(t) = \sum_{j=1}^{n} F_j(t)\mathbf{e}_j.$$

Then

$$\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{x}_j(t),$$

where $\mathbf{x}_j(t)$ is a particular solution of the simpler equation

$$\mathbf{x}'(t) = A\mathbf{x}(t) + f(t)\mathbf{c}, \quad f = F_j, \quad \mathbf{c} = \mathbf{e}_j.$$

An initial trial solution $\mathbf{x}(t)$ for $\mathbf{x}'(t) = A\mathbf{x}(t) + f(t)\mathbf{c}$ can be determined from the following **initial trial solution rule**:

> Assume $f(t)$ is a sum of atoms. Identify independent functions whose linear combinations give all derivatives of $f(t)$. Let the initial trial solution be a linear combination of these functions with undetermined vector coefficients $\{\mathbf{c}_j\}$.

In the well-known scalar case, the trial solution must be modified if its terms contain any portion of the general solution to the homogeneous equation. If $f(t)$ is a polynomial, then the *correction rule* for the initial trial solution is avoided by assuming the matrix A is invertible. This assumption means that $r = 0$ is not a root of $\det(A - rI) = 0$, which prevents the homogenous solution from having any polynomial terms.

The initial vector trial solution is substituted into the differential equation to find the undetermined coefficients $\{\mathbf{c}_j\}$, hence finding a particular solution.

Theorem 29 (Polynomial solutions)
Let $f(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!}$ be a polynomial of degree k. Assume A is an $n \times n$ constant invertible matrix. Then $\mathbf{u}' = A\mathbf{u} + f(t)\mathbf{c}$ has a polynomial solution $\mathbf{u}(t) = \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!}$ of degree k with vector coefficients $\{\mathbf{c}_j\}$ given by the relations

$$\mathbf{c}_j = -\sum_{i=j}^{k} p_i A^{j-i-1}\mathbf{c}, \quad 0 \le j \le k.$$

Theorem 30 (Polynomial $\times$ exponential solutions)
Let $g(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!}$ be a polynomial of degree k. Assume A is an $n \times n$ constant matrix and $B = A - aI$ is invertible. Then $\mathbf{u}' = A\mathbf{u} + e^{at} g(t)\mathbf{c}$ has a polynomial-exponential solution $\mathbf{u}(t) = e^{at} \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!}$ with vector coefficients $\{\mathbf{c}_j\}$ given by the relations

$$\mathbf{c}_j = -\sum_{i=j}^{k} p_i B^{j-i-1}\mathbf{c}, \quad 0 \le j \le k.$$

Proof of Theorem 29. Substitute $\mathbf{u}(t) = \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!}$ into the differential equation, then

$$\sum_{j=0}^{k-1} \mathbf{c}_{j+1} \frac{t^j}{j!} = A \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!} + \sum_{j=0}^{k} p_j \frac{t^j}{j!}\mathbf{c}.$$

Then terms on the right for $j = k$ must add to zero and the others match the left side coefficients of $t^j/j!$, giving the relations

$$A\mathbf{c}_k + p_k\mathbf{c} = \mathbf{0}, \quad \mathbf{c}_{j+1} = A\mathbf{c}_j + p_j\mathbf{c}.$$

Solving these relations recursively gives the formulas

$$\begin{array}{lcl} \mathbf{c}_k & = & -p_k A^{-1}\mathbf{c}, \\ \mathbf{c}_{k-1} & = & -\left(p_{k-1}A^{-1} + p_k A^{-2}\right)\mathbf{c}, \\ & \vdots & \\ \mathbf{c}_0 & = & -\left(p_0 A^{-1} + \cdots + p_k A^{-k-1}\right)\mathbf{c}. \end{array}$$

The relations above can be summarized by the formula

$$\mathbf{c}_j = -\sum_{i=j}^{k} p_i A^{j-i-1}\mathbf{c}, \quad 0 \le j \le k.$$

The calculation shows that if $\mathbf{u}(t) = \sum_{j=0}^{k} \mathbf{c}_j \frac{t^j}{j!}$ and $\mathbf{c}_j$ is given by the last formula, then $\mathbf{u}(t)$ substituted into the differential equation gives matching LHS and RHS. The proof is complete.

Proof of Theorem 30. Let $\mathbf{u}(t) = e^{at}\mathbf{v}(t)$. Then $\mathbf{u}' = A\mathbf{u} + e^{at}g(t)\mathbf{c}$ implies $\mathbf{v}' = (A - aI)\mathbf{v} + g(t)\mathbf{c}$. Apply Theorem 29 to $\mathbf{v}' = B\mathbf{v} + g(t)\mathbf{c}$. The proof is complete.