

11.5 The Eigenanalysis Method

The general solution $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ of the linear system

$$\mathbf{x}' = A\mathbf{x}$$

can be obtained entirely by eigenanalysis of the matrix A . A computationally useful case is when the $n \times n$ matrix A has n independent eigenpairs in its list of eigenpairs

$$(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), \dots, (\lambda_n, \mathbf{v}_n).$$

It is not required that the eigenvalues $\lambda_1, \dots, \lambda_n$ be distinct. The eigenvalues can be real or complex.

The Eigenanalysis Method for a 2×2 Matrix

Suppose that A is 2×2 real and has eigenpairs

$$(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2),$$

with $\mathbf{v}_1, \mathbf{v}_2$ independent. The eigenvalues λ_1, λ_2 can be both real. Also, they can be a complex conjugate pair $\lambda_1 = \bar{\lambda}_2 = a + ib$ with $b > 0$.

It will be shown that the general solution of $\mathbf{x}' = A\mathbf{x}$ can be written as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

The details:

$$\begin{aligned} \mathbf{x}' &= c_1 (e^{\lambda_1 t})' \mathbf{v}_1 + c_2 (e^{\lambda_2 t})' \mathbf{v}_2 && \text{Differentiate the formula for } \mathbf{x}. \\ &= c_1 e^{\lambda_1 t} \lambda_1 \mathbf{v}_1 + c_2 e^{\lambda_2 t} \lambda_2 \mathbf{v}_2 \\ &= c_1 e^{\lambda_1 t} A \mathbf{v}_1 + c_2 e^{\lambda_2 t} A \mathbf{v}_2 && \text{Use } \lambda_1 \mathbf{v}_1 = A \mathbf{v}_1, \lambda_2 \mathbf{v}_2 = A \mathbf{v}_2. \\ &= A (c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2) && \text{Factor } A \text{ left.} \\ &= A \mathbf{x} && \text{Definition of } \mathbf{x}. \end{aligned}$$

Let's rewrite the solution \mathbf{x} in the vector-matrix form

$$\mathbf{x}(t) = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Because eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ are assumed independent, then $\mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2)$ is invertible and setting $t = 0$ in the previous display gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2)^{-1} \mathbf{x}(0).$$

Because c_1, c_2 can be chosen to produce any initial condition $\mathbf{x}(0)$, then $\mathbf{x}(t)$ is the *general solution* of the system $\mathbf{x}' = A\mathbf{x}$.

The general solution expressed as $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ leads to the exponential matrix relation

$$e^{At} = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2)^{-1}.$$

The formula is immediately useful when the eigenpairs are real.

Complex conjugate eigenvalues. First, eigenpair $(\lambda_2, \mathbf{v}_2)$ is never computed or used, because $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ implies $A\bar{\mathbf{v}}_1 = \bar{\lambda}_1\bar{\mathbf{v}}_1$, which implies $\lambda_2 = \bar{\lambda}_1$ has eigenvector $\mathbf{v}_2 = \bar{\mathbf{v}}_1$.

If A is real, then e^{At} is real, and taking real parts across the formula for e^{At} will give a real formula. Due to the unpleasantness of the complex algebra, we will report the answer found, which is *real*, and then justify it with minimal use of complex numbers.

Define for eigenpair $(\lambda_1, \mathbf{v}_1)$ symbols a, b, P as follows:

$$\lambda_1 = a + ib, \quad b > 0, \quad P = \mathbf{aug}(\mathcal{R}e(\mathbf{v}_1), \mathcal{I}m(\mathbf{v}_1)).$$

Then

$$(1) \quad e^{At} = e^{at} P \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} P^{-1}.$$

Justification of (1). The formula is established by showing that the matrix $\Phi(t)$ on the right satisfies $\Phi(0) = I$ and $\Phi' = A\Phi$. Then by definition, $e^{At} = \Phi(t)$. For exposition, let

$$R(t) = e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}, \quad \Phi(t) = PR(t)P^{-1}.$$

The identity $\Phi(0) = I$ verified as follows.

$$\begin{aligned} \Phi(0) &= PR(0)P^{-1} \\ &= Pe^0IP^{-1} \\ &= I \end{aligned}$$

The expansion of eigenpair relation

$$A(\mathcal{R}e(\mathbf{v}_1) + i\mathcal{I}m(\mathbf{v}_1)) = (a + ib)(\mathcal{R}e(\mathbf{v}_1) + i\mathcal{I}m(\mathbf{v}_1))$$

into real and imaginary parts shows that

$$AP = P \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Then

$$\begin{aligned} \Phi'(t)\Phi^{-1}(t) &= PR'(t)P^{-1}PR^{-1}(t)P^{-1} \\ &= PR'(t)R^{-1}(t)P^{-1} \\ &= P \left(aI + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right) P^{-1} \\ &= P \begin{pmatrix} a & b \\ -b & a \end{pmatrix} P^{-1} \\ &= A \end{aligned}$$

The proof of $\Phi'(t) = A\Phi(t)$ is complete.

The formula for e^{At} implies that the general solution in this special case is

$$\mathbf{x}(t) = e^{at} \mathbf{aug}(\mathcal{R}e(\mathbf{v}_1), \mathcal{I}m(\mathbf{v}_1)) \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The values c_1, c_2 are related to the initial condition $\mathbf{x}(0)$ by the matrix identity

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{aug}(\mathcal{R}e(\mathbf{v}_1), \mathcal{I}m(\mathbf{v}_1))^{-1} \mathbf{x}(0).$$

The Eigenanalysis Method for a 3×3 Matrix

Suppose that A is 3×3 real and has eigenpairs

$$(\lambda_1, \mathbf{v}_1), \quad (\lambda_2, \mathbf{v}_2), \quad (\lambda_3, \mathbf{v}_3),$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ independent. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ can be all real. Also, there can be one real eigenvalue λ_3 and a complex conjugate pair of eigenvalues $\lambda_1 = \bar{\lambda}_2 = a + ib$ with $b > 0$.

The general solution of $\mathbf{x}' = A\mathbf{x}$ can be written as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + c_3 e^{\lambda_3 t} \mathbf{v}_3.$$

The details parallel the 2×2 details; they are left as an exercise for the reader.

The solution \mathbf{x} is written in vector-matrix form

$$\mathbf{x}(t) = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Because the three eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are assumed independent, then $\mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible. Setting $t = 0$ in the previous display gives

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)^{-1} \mathbf{x}(0).$$

Constants c_1, c_2, c_3 can be chosen to produce any initial condition $\mathbf{x}(0)$, therefore $\mathbf{x}(t)$ is the *general solution* of the 3×3 system $\mathbf{x}' = A\mathbf{x}$. There is a corresponding exponential matrix relation

$$e^{At} = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)^{-1}.$$

This formula is normally used when the eigenpairs are real. When there is a complex conjugate pair of eigenvalues $\lambda_1 = \bar{\lambda}_2 = a + ib$, $b > 0$, then as was shown in the 2×2 case it is possible to extract a real solution \mathbf{x} from the complex formula and report a real form for the exponential matrix:

$$e^{At} = P \begin{pmatrix} e^{at} \cos bt & e^{at} \sin bt & 0 \\ -e^{at} \sin bt & e^{at} \cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} P^{-1},$$

$$P = \mathbf{aug}(\mathcal{R}e(\mathbf{v}_1), \mathcal{I}m(\mathbf{v}_1), \mathbf{v}_3).$$

The Eigenanalysis Method for an $n \times n$ Matrix

The general solution formula and the formula for e^{At} generalize easily from the 2×2 and 3×3 cases to the general case of an $n \times n$ matrix.

Theorem 17 (The Eigenanalysis Method)

Let the $n \times n$ real matrix A have eigenpairs

$$(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), \dots, (\lambda_n, \mathbf{v}_n),$$

with n independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{v}_n e^{\lambda_n t}.$$

The vector-matrix form of the general solution is

$$\mathbf{x}(t) = \mathbf{aug}(\mathbf{v}_1, \dots, \mathbf{v}_n) \mathbf{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

This form is real provided all eigenvalues are real. A real form can be made from a complex form by following the example of a 3×3 matrix A . The plan is to list all complex eigenvalues first, in pairs, $\lambda_1, \bar{\lambda}_1, \dots, \lambda_p, \bar{\lambda}_p$. Then the real eigenvalues r_1, \dots, r_q are listed, $2p + q = n$. Define

$$P = \mathbf{aug}(\mathcal{R}e(\mathbf{v}_1), \mathcal{I}m(\mathbf{v}_1), \dots, \mathcal{R}e(\mathbf{v}_{2p-1}), \mathcal{I}m(\mathbf{v}_{2p-1}), \mathbf{v}_{2p+1}, \dots, \mathbf{v}_n),$$

$$R_\lambda(t) = e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}, \quad \text{where } \lambda = a + ib, \quad b > 0.$$

Then the real vector-matrix form of the general solution is

$$\mathbf{x}(t) = P \mathbf{diag}(R_{\lambda_1}(t), \dots, R_{\lambda_p}(t), e^{r_1 t}, \dots, e^{r_q t}) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

and

$$e^{At} = P \mathbf{diag}(R_{\lambda_1}(t), \dots, R_{\lambda_p}(t), e^{r_1 t}, \dots, e^{r_q t}) P^{-1}.$$

Spectral Theory Methods

The simplicity of Putzer's spectral method for computing e^{At} is appreciated, but we also recognize that the literature has an algorithm to compute e^{At} , devoid of differential equations, which is of fundamental importance in linear algebra. The parallel algorithm computes e^{At} directly from the eigenvalues λ_j of A and certain products of the nilpotent matrices $A - \lambda_j I$. Called **spectral formulas**, they can be implemented in a numerical laboratory or computer algebra system, in order to efficiently compute e^{At} , even in the case of multiple eigenvalues.

Theorem 18 (Computing e^{At} for Simple Eigenvalues)

Let the $n \times n$ matrix A have n simple eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly complex) and define constant matrices $\mathbf{Q}_1, \dots, \mathbf{Q}_n$ by the formulas

$$\mathbf{Q}_j = \prod_{i \neq j} \frac{A - \lambda_i I}{\lambda_j - \lambda_i}, \quad j = 1, \dots, n.$$

Then

$$e^{At} = e^{\lambda_1 t} \mathbf{Q}_1 + \dots + e^{\lambda_n t} \mathbf{Q}_n.$$

Theorem 19 (Computing e^{At} for Multiple Eigenvalues)

Let the $n \times n$ matrix A have k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of algebraic multiplicities m_1, \dots, m_k . Let $p(\lambda) = \det(A - \lambda I)$ and define polynomials $a_1(\lambda), \dots, a_k(\lambda)$ by the partial fraction identity

$$\frac{1}{p(\lambda)} = \frac{a_1(\lambda)}{(\lambda - \lambda_1)^{m_1}} + \dots + \frac{a_k(\lambda)}{(\lambda - \lambda_k)^{m_k}}.$$

Define constant matrices $\mathbf{Q}_1, \dots, \mathbf{Q}_k$ by the formulas

$$\mathbf{Q}_j = a_j(A) \prod_{i \neq j} (A - \lambda_i I)^{m_i}, \quad j = 1, \dots, k.$$

Then

$$(2) \quad e^{At} = \sum_{i=1}^k e^{\lambda_i t} \mathbf{Q}_i \sum_{j=0}^{m_i-1} (A - \lambda_i I)^j \frac{t^j}{j!}.$$

Proof: Let $\mathbf{N}_i = \mathbf{Q}_i(A - \lambda_i I)$, $1 \leq i \leq k$. We first prove

Lemma 1 (Properties)

1. $\mathbf{Q}_1 + \dots + \mathbf{Q}_k = I$,
2. $\mathbf{Q}_i \mathbf{Q}_i = \mathbf{Q}_i$,
3. $\mathbf{Q}_i \mathbf{Q}_j = \mathbf{0}$ for $i \neq j$,
4. $\mathbf{N}_i \mathbf{N}_j = \mathbf{0}$ for $i \neq j$,
5. $\mathbf{N}_i^{m_i} = \mathbf{0}$,
6. $A = \sum_{i=1}^k (\lambda_i \mathbf{Q}_i + \mathbf{N}_i)$.

The proof of **1** follows from clearing fractions in the partial fraction expansion of $1/p(\lambda)$:

$$1 = \sum_{i=1}^k a_i(\lambda) \frac{p(\lambda)}{(\lambda - \lambda_i)^{m_i}}.$$

The **projection property 2** follows by multiplication of identity **1** by \mathbf{Q}_i and then using **2**.

The proof of **3** starts by observing that \mathbf{Q}_i and \mathbf{Q}_j together contain all the factors of $p(A)$, therefore $\mathbf{Q}_i \mathbf{Q}_j = q(A)p(A)$ for some polynomial q . The Cayley-Hamilton theorem $p(A) = \mathbf{0}$ finishes the proof.

To prove **4**, write $\mathbf{N}_i \mathbf{N}_j = (A - \lambda_i I)(A - \lambda_j I) \mathbf{Q}_i \mathbf{Q}_j$ and apply **3**.

To prove **5**, use $\mathbf{Q}_i^{m_i} = \mathbf{Q}_i$ (from **2**) to write $\mathbf{N}_i^{m_i} = (A - \lambda_i I)^{m_i} \mathbf{Q}_i = p(A) = \mathbf{0}$.

To prove **6**, multiply **1** by A and rearrange as follows:

$$\begin{aligned} A &= \sum_{i=1}^k A \mathbf{Q}_i \\ &= \sum_{i=1}^k \lambda_i \mathbf{Q}_i + (A - \lambda_i I) \mathbf{Q}_i \\ &= \sum_{i=1}^k \lambda_i \mathbf{Q}_i + \mathbf{N}_i \end{aligned}$$

To prove (2), multiply **1** by e^{At} and compute as follows:

$$\begin{aligned} e^{At} &= \sum_{i=1}^k \mathbf{Q}_i e^{At} \\ &= \sum_{i=1}^k \mathbf{Q}_i e^{\lambda_i t + (A - \lambda_i I)t} \\ &= \sum_{i=1}^k \mathbf{Q}_i e^{\lambda_i t} e^{(A - \lambda_i I)t} \\ &= \sum_{i=1}^k \mathbf{Q}_i e^{\lambda_i t} e^{\mathbf{N}_i t} \\ &= \sum_{i=1}^k \mathbf{Q}_i e^{\lambda_i t} e^{\mathbf{N}_i t} \\ &= \sum_{i=1}^k \mathbf{Q}_i e^{\lambda_i t} \sum_{j=0}^{m_i-1} \frac{(A - \lambda_i I)^j t^j}{j!} \end{aligned}$$

Solving Planar Systems $\mathbf{x}'(t) = A\mathbf{x}(t)$

A 2×2 real system $\mathbf{x}'(t) = A\mathbf{x}(t)$ can be solved in terms of the roots of the characteristic equation $\det(A - \lambda I) = 0$ and the real matrix A .

Theorem 20 (Planar System, Putzer's Spectral Formula)

Consider the real planar system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution $\mathbf{x}(t)$ is given by the formula

$$\mathbf{x}(t) = e^{At} \mathbf{x}(0)$$

where the 2×2 exponential matrix e^{At} is given as follows.

$$\text{Real } \lambda_1 \neq \lambda_2 \quad e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} (A - \lambda_1 I).$$

$$\text{Real } \lambda_1 = \lambda_2 \quad e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t} (A - \lambda_1 I).$$

$$\begin{aligned} \text{Complex } \lambda_1 = \bar{\lambda}_2, \\ \lambda_1 = a + bi, b > 0 \end{aligned} \quad e^{At} = e^{at} \cos bt I + \frac{e^{at} \sin(bt)}{b} (A - aI).$$

Proof: The formulas are from Putzer's algorithm, or equivalently, from the spectral formulas, with rearranged terms. The complex case is formally the real part of the distinct root case when $\lambda_2 = \bar{\lambda}_1$. The **spectral formula** is the analog of the second order equation formulas, Theorem 45 in Chapter 5.

Illustrations. Typical cases are represented by the following 2×2 matrices A , which correspond to roots λ_1, λ_2 of the characteristic equation $\det(A - \lambda I) = 0$ which are real distinct, real double or complex conjugate. The solution $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ is given here in two forms, by writing e^{At} using **1** a **spectral formula** and **2** Putzer's **spectral formula**.

$$\lambda_1 = 5, \lambda_2 = 2$$

$$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$$

Real distinct roots.

$$\mathbf{1} \quad e^{At} = \frac{e^{5t}}{3} \begin{pmatrix} -3 & 3 \\ -6 & 6 \end{pmatrix} + \frac{e^{2t}}{-3} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

$$\mathbf{2} \quad e^{At} = e^{5t}I + \frac{e^{2t} - e^{5t}}{2 - 5} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = 3$$

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

Real double root.

$$\mathbf{1} \quad e^{At} = e^{3t} \left(I + t \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right)$$

$$\mathbf{2} \quad e^{At} = e^{3t}I + te^{3t} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\lambda_1 = \bar{\lambda}_2 = 2 + 3i$$

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

Complex conjugate roots.

$$\mathbf{1} \quad e^{At} = 2 \operatorname{Re} \left(\frac{e^{2t+3it}}{2(3i)} \begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \right)$$

$$\mathbf{2} \quad e^{At} = e^{2t} \cos 3tI + \frac{e^{2t} \sin 3t}{3} \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

The complex example is typical for real $n \times n$ matrices A with a complex conjugate pair of eigenvalues $\lambda_1 = \bar{\lambda}_2$. Then $\mathbf{Q}_2 = \bar{\mathbf{Q}}_1$. The result is that λ_2 is not used and we write instead a simpler expression using the college algebra equality $z + \bar{z} = 2 \operatorname{Re}(z)$:

$$e^{\lambda_1 t} \mathbf{Q}_1 + e^{\lambda_2 t} \mathbf{Q}_2 = 2 \operatorname{Re} \left(e^{\lambda_1 t} \mathbf{Q}_1 \right).$$

This observation explains why e^{At} is real when A is real, by pairing complex conjugate eigenvalues in the spectral formula.