Classification of Phase Portraits at Equilibria for
\[ \vec{u}'(t) = \vec{f}(\vec{u}(t)) \]

- Transfer of Local Linearized Phase Portrait
- Transfer of Local Linearized Stability
- How to Classify Linear Equilibria
- Justification of the Classification Method
- Three Examples
  - Spiral-saddle Example
  - Center-saddle Example
  - Node-saddle Example
Transfer of Local Linearized Phase Portrait

THEOREM.
Let \( \vec{u}_0 \) be an equilibrium point of the nonlinear dynamical system

\[
\vec{u}'(t) = \vec{f}(\vec{u}(t)).
\]

Assume the Jacobian of \( \vec{f}(\vec{u}) \) at \( \vec{u} = \vec{u}_0 \) is matrix \( A \) and \( \vec{u}'(t) = A\vec{u}(t) \) has linear classification \text{saddle, node, center} \ or \text{spiral} at its equilibrium point \((0, 0)\).

Then the nonlinear system \( \vec{u}'(t) = \vec{f}(\vec{u}(t)) \) at equilibrium point \( \vec{u} = \vec{u}_0 \) has the same classification, with the following exceptions:

If the linear classification at \((0, 0)\) for \( \vec{u}'(t) = A\vec{u}(t) \) is a node or a center, then the nonlinear classification at \( \vec{u} = \vec{u}_0 \) might be a spiral.

The exceptions in terms of roots of the characteristic equation: \( \lambda_1 = \lambda_2 \) (real equal roots) and \( \lambda_1 = \lambda_2 = bi \) \((b > 0\), purely complex roots\).
Transfer of Local Linearized Stability

THEOREM.
Let $\mathbf{u}_0$ be an equilibrium point of the nonlinear dynamical system

$$\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}(t)).$$

Assume the Jacobian of $\mathbf{f}(\mathbf{u})$ at $\mathbf{u} = \mathbf{u}_0$ is matrix $A$. Then the nonlinear system $\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}(t))$ at $\mathbf{u} = \mathbf{u}_0$ has the same stability as $\mathbf{u}'(t) = A\mathbf{u}(t)$ with the following exception:

If the linear classification at $(0, 0)$ for $\mathbf{u}'(t) = A\mathbf{u}(t)$ is a center, then the nonlinear classification at $\mathbf{u} = \mathbf{u}_0$ might be either stable or unstable.
How to Classify Linear Equilibria

- Assume the linear system is $2 \times 2$, $\vec{u}' = A\vec{u}$.
- Compute the roots $\lambda_1, \lambda_2$ of the characteristic equation of $A$.
- Find the atoms $A_1(t), A_2(t)$ for these two roots.
- If the atoms have sine and cosine factors, then a rotation is implied and the classification is either a **center** or **spiral**. Pure harmonic atoms [no exponentials] imply a center, otherwise it’s a spiral.
- If the atoms are exponentials, then the classification is a non-rotation, a **node** or **saddle**. Take limits of the atoms at $t = \infty$ and also $t = -\infty$. If one limit answer is $A_1 = A_2 = 0$, then it’s a node, otherwise it’s a saddle.
Justification of the Classification Method

The Cayley-Hamilton-Ziebur theorem implies that the general solution of

$$\ddot{\mathbf{u}} = A\dot{\mathbf{u}}$$

is the equation

$$\mathbf{u}(t) = A_1(t)\mathbf{d}_1 + A_2(t)\mathbf{d}_2$$

where $A_1$, $A_2$ are the atoms corresponding to the roots $\lambda_1, \lambda_2$ of the characteristic equation of $A$. Although $\mathbf{d}_1, \mathbf{d}_2$ are not arbitrary, the classification only depends on the roots and hence only on the atoms. We construct examples of the behavior by choosing $\mathbf{d}_1, \mathbf{d}_2$, for example,

$$\mathbf{d}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

If the atoms were $\cos t$, $\sin t$, then the solution by C-H-Z would be $x = \cos t$, $y = \sin t$. Analysis of the trajectory shows a circle, hence we expect a center at $(0, 0)$. Similar examples can be invented for the other cases of a spiral, saddle, or node, by considering possible pairs of atoms.
Three Examples

Consider the nonlinear systems and selected equilibrium points. The third example has infinitely many equilibria.

**Spiral–Saddle**
\[
\begin{align*}
x' &= x + y, \\
y' &= 1 - x^2.
\end{align*}
\]
Equilibria \((1, -1), (-1, 1)\)

**Center–Saddle**
\[
\begin{align*}
x' &= y, \\
y' &= -20x + 5x^3.
\end{align*}
\]
Equilibria \((0, 0), (2, 0), (-2, 0)\)

**Node–Saddle**
\[
\begin{align*}
x' &= 3 \sin(x) + y, \\
y' &= \sin(x) + 2y.
\end{align*}
\]
Equilibria \((2\pi, 0), (\pi, 0)\)
Spiral-saddle Example

The nonlinear function and Jacobian are

$$\vec{f}(x, y) = \begin{pmatrix} x + y \\ 1 - x^2 \end{pmatrix}, \quad A(x, y) = \begin{pmatrix} 1 & 1 \\ -2x & 0 \end{pmatrix}.$$ 

Then $A(1, -1) = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$ and $A(-1, 1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$.

- The characteristic equations are $\lambda^2 - \lambda + 2 = 0$ and $\lambda^2 - \lambda - 2 = 0$ with roots $\frac{1}{2} \pm \frac{1}{2} \sqrt{7}i$ and $2, -1$, respectively.

- The atoms for $A(1, -1)$ are $e^{t/2} \cos(\sqrt{7}t/2), e^{t/2} \sin(\sqrt{7}t/2)$. Rotation implies a center or spiral. No pure harmonics, so it’s a spiral. The limit at $t = -\infty$ is zero for both atoms, so it’s stable at minus infinity, implying unstable at infinity.

- The atoms for $A(-1, 1)$ are $e^{2t}, e^{-t}$. No rotation implies a node or saddle. Neither the limit at infinity nor at minus infinity gives zero, so it’s a saddle.
Center-saddle Example

The nonlinear function and Jacobian are

\[ \vec{f}(x, y) = \begin{pmatrix} y \\ -20x + 5x^3 \end{pmatrix}, \quad A(x, y) = \begin{pmatrix} 0 & 1 \\ -20 + 15x^2 & 0 \end{pmatrix}. \]

Then \( A(0, 0) = \begin{pmatrix} 0 & 1 \\ -20 & 0 \end{pmatrix} \) and \( A(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ 40 & 0 \end{pmatrix}. \)

- The characteristic equations are \( \lambda^2 + 20 = 0 \) and \( \lambda^2 - 40 = 0 \) with roots \( \pm \sqrt{20i} \) and \( \pm \sqrt{40} \), respectively.

- The atoms for \( A(0, 0) \) are \( \cos(\sqrt{20}t), \sin(\sqrt{20}t) \). Rotation implies a center or spiral. The atoms are pure harmonics, so it’s a center. The nonlinear system can be a center or a spiral and either stable or unstable. The issue is decided by a computer algebra system to be a center.

- The atoms for \( A(\pm 2, 0) \) are \( e^{bt}, e^{-bt} \), where \( b = \sqrt{40} \). No rotation implies a node or saddle. Neither the limit at infinity nor at minus infinity gives zero, so it’s a saddle.
The nonlinear function and Jacobian are

\[
\vec{f}(x, y) = \begin{pmatrix} 3 \sin x + y \\ \sin x + 2y \end{pmatrix}, \quad A(x, y) = \begin{pmatrix} 3 \cos x & 1 \\ \cos x & 2 \end{pmatrix}.
\]

Then \( A(2\pi, 0) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \) and \( A(\pi, 0) = \begin{pmatrix} -3 & 1 \\ -1 & 2 \end{pmatrix} \).

• The characteristic equations are \( \lambda^2 - 5\lambda + 5 = 0 \) and \( \lambda^2 + \lambda - 5 = 0 \) with roots \( \frac{1}{2}(5 \pm \sqrt{5}) = 3.6, 1.38 \) and \( \frac{1}{2}(-1 \pm \sqrt{21}) = 1.79, -2.79 \), respectively.

• The atoms for \( A(2\pi, 0) \) are \( e^{at}, e^{bt} \) with \( a > 0, b > 0 \). No rotation implies a node or saddle. The atoms limit to zero at \( t = -\infty \), so one end is stable, which eliminates the saddle. It’s a node, unstable at infinity.

• The atoms for \( A(\pi, 0) \) are \( e^{at}, e^{bt} \), where \( a > 0 \) and \( b < 0 \). No rotation implies a node or saddle. Neither the limit at infinity nor at minus infinity gives zero, so it’s a saddle.

• The two classifications and their stability transfers to the nonlinear system. The only case when a node does not automatically transfer is the case of equal roots.