Three Examples. It is possible to solve a variety of differential equations without reading a differential equations textbook:

Growth-Decay $\quad \dfrac{dA}{dt} = kA(t),\ A(0) = A_0$

$$\boxed{A(t) = A_0 e^{kt}}$$

Newton Cooling $\quad \dfrac{du}{dt} = -h(u(t) - u_1),\ u(0) = u_0$

$$\boxed{u(t) = u_1 + (u_0 - u_1)e^{-ht}}$$

Verhulst Logistic $\quad \dfrac{dP}{dt} = (a - bP(t))P(t),\ P(0) = P_0$

$$\boxed{P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}}$$

Like the multiplication tables in elementary school, these models and their solution formulas should be *memorized*, in order to form a foundation of intuition for all differential equation theory. The last two solutions are derived from the growth-decay equation by variable changes: $A(t) = u(t) - u_1$ and $A(t) = P(t)/(a - bP(t))$.

Theorem 1 (Recipe for Second-Order Constant Equations)
Let $a \neq 0$, b and c be real constants. Each solution $y(x)$ of the constant-coefficient second-order differential equation $ay''(x) + by'(x) + cy(x) = 0$ is given for some constants c_1, c_2 by the expression

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where $y_1(x)$, $y_2(x)$ are two special solutions of $ay'' + by' + cy = 0$ defined by the following **recipe**:

Let r_1, r_2 be the two roots of the **characteristic equation** $ar^2 + br + c = 0$, real or complex. If complex, then let $r_1 = \overline{r_2} = \alpha + i\beta$ with $\beta > 0$. Then y_1, y_2 are given by the following three cases, organized by the sign of the discriminant $\mathcal{D} = b^2 - 4ac$:

1. $\mathcal{D} > 0$ (Real distinct roots)	$y_1 = e^{r_1 x}$	$y_2 = e^{r_2 x}$.
2. $\mathcal{D} = 0$ (Real equal roots)	$y_1 = e^{r_1 x}$	$y_2 = xe^{r_1 x}$.
3. $\mathcal{D} < 0$ (Complex roots)	$y_1 = e^{\alpha x}\cos(\beta x)$	$y_2 = e^{\alpha x}\sin(\beta x)$.

A **general solution** is an expression that represents all solutions of the differential equation. The term **recipe** means that the general solution can be written out at very high speed with no justification required. Some typical examples (c_1, c_2 = arbitrary constants):

Distinct real roots $\lambda_1 = 5,\ \lambda_2 = 2$	Equal real roots $\lambda_1 = 3,\ \lambda_2 = 3$	Complex conjugate roots $\lambda_1 = \overline{\lambda_2} = 2 + 3i$
$y = c_1 e^{5x} + c_2 e^{2x}$	$y = c_1 e^{3x} + c_2 x e^{3x}$	$y = c_1 e^{2x} \cos 3x + c_2 e^{2x} \sin 3x$

Solving planar systems $\mathbf{x}'(t) = A\mathbf{x}(t)$. A 2×2 real system $\mathbf{x}'(t) = A\mathbf{x}(t)$ can be solved in terms of the roots of the characteristic equation $\det(A - \lambda I) = 0$ and the real matrix A. The practical formulas are the analog of the *recipe* for second order equations with constant coefficients.

Theorem 2 (Planar Constant-Coefficient Linear system)
Consider the real planar system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution $\vec{\mathbf{x}}(t)$ is given by the formulae

Distinct real roots $\lambda_1 \neq \lambda_2$	Equal real roots $\lambda_1 = \lambda_2$	Complex conjugate roots $\lambda_1 = \overline{\lambda_2} = \alpha + i\beta,\ \beta > 0$
$\vec{\mathbf{x}} = \vec{\mathbf{u}}_1 e^{\lambda_1 t} + \vec{\mathbf{u}}_2 e^{\lambda_2 t}$	$\vec{\mathbf{x}} = \vec{\mathbf{u}}_1 e^{\lambda_1 t} + \vec{\mathbf{u}}_2 t e^{\lambda_1 t}$	$\vec{\mathbf{x}} = \vec{\mathbf{u}}_1 e^{\alpha t} \cos \beta t + \vec{\mathbf{u}}_2 e^{\alpha t} \sin \beta t$
$\vec{\mathbf{u}}_1 = \dfrac{A - \lambda_2 I}{\lambda_1 - \lambda_2} \vec{\mathbf{x}}(0)$	$\vec{\mathbf{u}}_1 = \vec{\mathbf{x}}(0)$	$\vec{\mathbf{u}}_1 = \vec{\mathbf{x}}(0)$
$\vec{\mathbf{u}}_2 = \dfrac{A - \lambda_1 I}{\lambda_2 - \lambda_1} \vec{\mathbf{x}}(0)$	$\vec{\mathbf{u}}_2 = (A - \lambda_1 I)\vec{\mathbf{x}}(0)$	$\vec{\mathbf{u}}_2 = \dfrac{1}{\beta}(A - \alpha I)\vec{\mathbf{x}}(0)$

Illustrations. Typical cases are represented by the following 2×2 matrices A:

$\lambda_1 = 5,\ \lambda_2 = 2$ Real distinct roots.

$$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix} \qquad \vec{\mathbf{x}}(t) = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \vec{\mathbf{x}}(0) e^{5t} + \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \vec{\mathbf{x}}(0) e^{2t}.$$

$\lambda_1 = \lambda_2 = 3$ Real double root.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \qquad \vec{\mathbf{x}}(t) = \vec{\mathbf{x}}(0) e^{3t} + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \vec{\mathbf{x}}(0) t e^{3t}.$$

$\lambda_1 = \overline{\lambda_2} = 2 + 3i$ Complex conjugate roots.

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \qquad \vec{\mathbf{x}}(t) = \vec{\mathbf{x}}(0) e^{2t} \cos 3t + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{\mathbf{x}}(0) e^{2t} \sin 3t$$

$$= \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix} \vec{\mathbf{x}}(0) e^{2t}.$$