# 2.8 Science and Engineering Applications

Assembled here are some classical applications of first order differential equations to problems of science and engineering.



## Draining a Tank

Investigated here is a tank of water with orifice at the bottom emptying due to gravity; see Figure 7. The analysis applies to tanks of any geometrical shape.

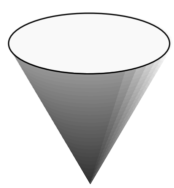


**Figure 7. Draining a tank.**
A tank empties from an orifice at the bottom. The fluid fills the tank to height $y$ above the orifice, and it drains due to gravity.

Evangelista Torricelli (1608-1647), inventor of the **barometer**, investigated this physical problem using Newton's laws, obtaining the result in Lemma 8, proof on page 158.

**Lemma 8 (Torricelli)** A droplet falling freely from height $h$ in a gravitational field with constant $g$ arrives at the orifice with speed $\sqrt{2gh}$.

**Tank Geometry**. A simple but useful tank geometry can be constructed using *washers* of area $A(y)$, where $y$ is the height above the orifice; see Figure 8.

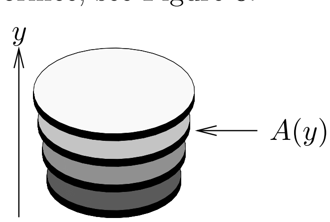



**Figure 8. A tank constructed from washers.**

Then the method of cross-sections in calculus implies that the *volume* $V(h)$ of the tank at height $h$ is given by

$$(1) \qquad V(h) = \int_0^h A(y)dy, \quad \frac{dV}{dh} = A(h).$$

**Torricelli's Equation**. Torricelli's lemma applied to the tank fluid height $y(t)$ at time $t$ implies, by matching drain rates at the orifice (see *Technical Details* page 158), that

$$(2) \qquad \frac{d}{dt}\left(V(y(t))\right) = -k\sqrt{y(t)}$$

for some proportionality constant $k > 0$. The *chain rule* gives the separable differential equation $V'(y(t))y'(t) = -k\sqrt{y(t)}$, or equivalently (see page 158), in terms of the **cross-sectional area** $A(y) = V'(y)$,

$$(3) \qquad y'(t) = -k\frac{\sqrt{y(t)}}{A(y(t))}.$$

Typical of the physical literature, the requirement $y(t) \ge 0$ is omitted in the model, but assumed implicitly. The model itself **exhibits non-uniqueness**: the tank can be drained hours ago or at instant $t = 0$ and result still in the solution $y(t) = 0$, interpreted as fluid height zero.

## Stefan's Law

Heat energy can be transferred by **conduction**, **convection** or **radiation**. The following illustrations suffice to distinguish the three types of heat transfer.

**Conduction.** A soup spoon handle gains heat from the soup by exchange of kinetic energy at a molecular level.

**Convection.** A hot water radiator heats a room largely by *convection currents*, which move heated air upwards and denser cold air downwards to the radiator. In linear applications, **Newton's cooling law** applies.

**Radiation.** A car seat heated by the sun gets the heat energy from electromagnetic waves, which carry energy from the sun to the earth.

The rate at which an object emits or absorbs **radiant energy** is given by **Stefan's radiation law**

$$P = kT^4.$$

The symbol $P$ is the power in watts (joules per second), $k$ is a constant proportional to the surface area of the object and $T$ is the temperature of the object in degrees Kelvin. Use $K = C + 273.15$ to convert Celsius to Kelvin. The constant $k$ in Stefan's law is decomposed as $k = \sigma A \mathcal{E}$. Here, $\sigma = 5.6696 \times 10^{-8} K^{-4}$ Watts per square meter ($K$=Kelvin), $A$ is the surface area of the body in square meters and $\mathcal{E}$ is the **emissivity**, which is a constant between 0 and 1 depending on properties of the surface.

**Constant room temperature**. Suppose that a person with skin temperature $T$ Kelvin sits unclothed in a room in which the thermometer reads $T_0$ Kelvin. The net heat flux $P_{\text{net}}$ in joules per second (watts) is given by

$$P_{\text{net}} = k(T^4 - T_0^4). \tag{4}$$

If $T$ and $T_0$ are constant, then $Q = kt(T^4 - T_0^4)$ can be used to estimate the total heat loss or gain in joules for a time period $t$. To illustrate, if the wall thermometer reads 20° Celsius, then $T_0 = 20 + 273.15$. Assume $A = 1.5$ square meters, $\mathcal{E} = 0.9$ and skin temperature 33° Celsius or $T = 33 + 273.15$. The total heat loss in 10 minutes is $Q = (10(60))(5.6696 \times 10^{-8})(1.5)(0.9)(305.15^4 - 293.15^4) = 64282$ joules. Over one hour, the total heat radiated is approximately $385,691$ joules, which is close to the total energy provided by a 6 ounce soft drink.[4]

**Time-varying room temperature**. Suppose that a person with skin temperature $T$ degrees Kelvin sits unclothed in a room. Assume the thermometer initially reads 15° Celsius and then rises to 24° Celsius after $t_1$ seconds. The function $T_0(t)$ has values $T_0(0) = 15 + 273.15$ and $T_0(t_1) = 24 + 273.15$. In a possible physical setting, $T_0(t)$ reflects the reaction to the heating and cooling system, which is generally oscillatory about the thermostat setting. If the thermostat is off, then it is reasonable to assume a linear model $T_0(t) = at + b$, with $a = (T_0(t_1) - T_0(0))/t_1$, $b = T_0(0)$.

To compute the total heat radiated from the person's skin, we use the time-varying equation

$$\frac{dQ}{dt} = k(T^4 - T_0(t)^4). \tag{5}$$

The solution to (5) with $Q(0) = 0$ is formally given by the quadrature formula

$$Q(t) = k \int_0^t (T^4 - T_0(r)^4) dr. \tag{6}$$

[4]American soft drinks are packaged in 12-ounce cans, twice the quantity cited. One calorie is *defined* to be 4.186 joules and one food **Calorie** is 1000 calories (a kilo-calorie) or 4186 joules. A boxed apple juice is about 6 ounces or 0.2 liters. Juice provides about 400 thousand joules in 0.2 liters. Product labels advertising 96 Calories mean 96 kilo-calories; it converts to $96(1000)(4.186) = 401,856$ joules.

For the case of a linear model $T_0(t) = at + b$, the total number of joules radiated from the person's skin is found by integrating (6), giving

$$Q(t_1) = kT^4 t_1 + k\frac{b^5 - (at_1 + b)^5}{5a}.$$

## Tsunami

A **seismic sea wave** due to an earthquake under the sea or some other natural event, called a **tsunami**, creates a wave on the surface of the ocean. The wave may have a height of less than 1 meter. These waves can have a very large wavelength, up to several hundred miles, depending upon the depth of the water where they were formed. The period is often more than one hour with wave velocity near 700 kilometers per hour. These waves contain a huge amount of energy. Their height increases as they crash upon the shore, sometimes to 30 meters high or more, depending upon water depth and underwater surface features. In the year 1737, a wave estimated to be 64 meters high hit Cape Lopatka, Kamchatka, in northeast Russia. The largest Tsunami ever recorded occurred in July of 1958 in Lituya Bay, Alaska, when a huge rock and ice fall caused water to surge up to 500 meters. For additional material on earthquakes, see page 731. For the Sumatra and Chile earthquakes, and resultant Tsunamis, see page 734.

**Wave shape**. A simplistic model for the shape $y(x)$ of a tsunami in the open sea is the differential equation

$$(y')^2 = 4y^2 - 2y^3. \tag{7}$$

This equation gives the *profile* $y(x)$ of one side of the 3D-wave, by cutting the 3D object with an $xy$-plane.

**Equilibrium solutions**. They are $y = 0$ and $y = 2$, corresponding to **no wave** and a **wall of water** 2 units above the ocean surface. There are *no solutions* for $y > 2$, because the two sides of (7) have in this case different signs.

**Non-equilibrium solutions**. They are given by

$$y(x) = 2 - 2\tanh^2(x + c). \tag{8}$$

The initial height of the wave is related to the parameter $c$ by $y(0) = 2 - 2\tanh^2(c)$. Only initial heights $0 < y(0) < 2$ are physically significant. Due to the property $\lim_{u\to\infty} \tanh(u) = 1$ of the hyperbolic tangent, the wave height starts at $y(0)$ and quickly decreases to zero (sea level), as is evident from Figure 9.

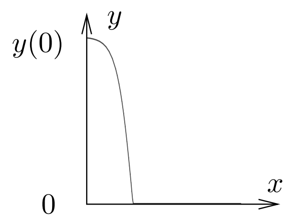



**Figure 9. A tsunami profile.**

**Non-uniqueness**. When $y(x_0) = 2$ for some $x = x_0$, then also $y'(x_0) = 0$, and this allows non-uniqueness of the solution $y$. An interesting solution different from equation (8) is the piecewise function

$$y(x) = \begin{cases} 2 - 2\tanh^2(x - x_0) & x > x_0, \\ 2 & x \le x_0. \end{cases} \tag{9}$$

This shape is an approximation to observed waves, in which the usual crest of the wave has been flattened. See Figure 12 on page 157.

## Gompertz Tumor Equation

Researchers in tumor growth have shown that for some solid tumors the volume $V(t)$ of dividing cells at time $t$ approaches a limiting volume $M$, even though the tumor volume may increase by 1000 fold. Gompertz is credited with an equation which fits the growth cycle of some solid tumors; the **Gompertzian relation** is

$$V(t) = V_0 e^{\frac{a}{b}\left(1-e^{-bt}\right)}. \tag{10}$$

The relation says that the doubling time for the total solid tumor volume *increases with time*. In contrast to a simple exponential model, which has a fixed doubling time and no volume limit, the limiting volume in the Gompertz model (10) is $M = V_0 e^{a/b}$.

**Experts suggest** to verify from Gompertz's relation (10) the formula

$$V' = ae^{-bt}V,$$

and then use this differential equation to argue why the tumor volume $V$ approaches a limiting value $M$ with a necrotic core; see *Technical Details for* (11), page 158.

**A different approach** is to make the substitution $y = V/V_0$ to obtain the differential equation

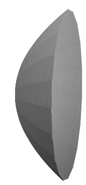
$$y' = (a - b\ln y)y, \tag{11}$$

which is almost a logistic equation, sometimes called the **Gompertz equation**. For details, see page 158. In analogy with logistic theory, low volume tumors should grow exponentially with rate $a$ and then slow down like a population that is approaching the carrying capacity.

The exact mechanism for the slowing of tumor growth can be debated. One view is that the number of reproductive cells is related to available oxygen and nutrients present only near the surface of the tumor, hence this number decreases with time as the necrotic core grows in size.

## Parabolic Mirror

Overhead projectors might use a high-intensity lamp located near a silvered reflector to provide a nearly parallel light source of high brightness. It is called a **parabolic mirror** because the surface of revolution is formed from a parabola, a fact which will be justified below.



The requirement is a shape $x = g(y)$ such that a light beam emanating from $C(0,0)$ reflects at point on the curve into a second beam parallel to the $x$-axis; see Figure 10. The **optical law of reflection** implies that the angle of incidence equals the angle of reflection, the straight reference line being the tangent to the curve $x = g(y)$.

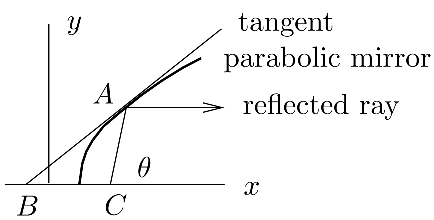



**Figure 10. A parabolic mirror.**

Symmetry suggests the restriction $y \geq 0$ will suffice to determine the shape. The assumption $y(0) = 1$ determines the $y$-axis scale.

The mirror shape $x = g(y)$ is shown in *Technical Details* page 159 to satisfy

$$\frac{dx}{dy} = \frac{x + \sqrt{x^2 + y^2}}{y}, \quad x(1) = 0. \tag{12}$$

This equation is equivalent for $y > 0$ to the separable equation $du/dy = \sqrt{u^2 + 1}$, $u(1) = 0$; see *Technical Details* page 159. Solving the separable equation (see page 159) gives the *parabola*

$$2x + 1 = y^2. \tag{13}$$

## Logarithmic Spiral

The polar curve

$$r = r_0 e^{k\theta} \tag{14}$$

is called a **logarithmic spiral**. In equation (14), symbols $r$, $\theta$ are polar variables and $r_0$, $k$ are constants. It will be shown that a logarithmic spiral has the following geometric characterization.

> A logarithmic spiral cuts each radial line from the origin at a constant angle.

The background required is the polar coordinate calculus formula

$$\tan(\alpha - \theta) = r\,\frac{d\theta}{dr} \tag{15}$$

where $\alpha$ is the angle between the $x$-axis and the tangent line at $(r, \theta)$; see Technical Details page 160. The angle $\alpha$ can also be defined from the calculus formula $\tan\alpha = dy/dx$.

The angle $\phi$ which a polar curve cuts a radial line is $\phi = \alpha - \theta$. By equation (15), the polar curve must satisfy the polar differential equation

$$r\frac{d\theta}{dr} = \frac{1}{k}$$

for constant $k = 1/\tan\phi$. This differential equation is separable with separated form

$$k d\theta = \frac{dr}{r}.$$

Solving gives $k\theta = \ln r + c$ or equivalently $r = r_0 e^{k\theta}$, for $c = -\ln r_0$. Hence equation (14) holds. All steps are reversible, therefore a logarithmic spiral is characterized by the geometrical description given above.

## Examples

39 **Example (Conical Tank)** A conical tank with $xy$-projection given in Figure 11 is realized by rotation about the $y$-axis. An orifice at $x = y = 0$ is created at time $t = 0$. Find an approximation for the drain time and the time to empty the tank to half-volume, given 10% drains in 20 seconds.

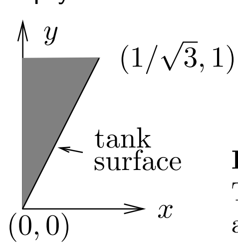



**Figure 11. Conical tank $xy$-projection.**
The tank is obtained by rotation of the shaded triangle about the $y$-axis. The cone has height 1.

**Solution**: The answers are approximately 238 seconds and 104 seconds. The incorrect drain time estimate of ten times the given 20 seconds is wrong by 19 percent. Doubling the half-volume time to find the drain time is equally invalid (both 200 and 208 are incorrect).

**Tank cross-section** $A(y)$. From Figure 11, the line segment along the tank surface has equation $y = \sqrt{3}x$; the equation was found from the two points $(0,0)$ and $(1/\sqrt{3},1)$ using the point-slope form of a line. A washer then has area $A(y) = \pi x^2$ or $A(y) = \pi y^2/3$.

**Tank half-volume** $V_h$. The half-volume is given by

$$\begin{aligned} V_h &= \frac{1}{2}V(1) && \text{Full volume is } V(1). \\ &= \frac{1}{2}\int_0^1 A(y)dy && \text{Apply } V(h) = \textstyle\int_0^h A(y)dy. \\ &= \frac{\pi}{18} && \text{Evaluate integral, } A(y) = \pi y^2/3. \end{aligned}$$

**Torricelli's equation**. The differential equation (3) becomes

$$(16) \qquad y'(t) = -\frac{3k}{\pi\sqrt{y^3(t)}}, \quad y(0) = 1,$$

with $k$ to be determined. The solution by separation of variables is

$$(17) \qquad y(t) = \left(1 - \frac{15k}{2\pi}\,t\right)^{2/5}.$$

The details:

$$\begin{aligned} y^{3/2}y' &= -\frac{3k}{\pi} && \text{Separated form.} \\ \frac{2}{5}y^{5/2} &= -\frac{3kt}{\pi} + C && \text{Integrate both sides.} \\ y^{5/2} &= -\frac{15kt}{2\pi} + 1 && \text{Isolate } y\text{, then use } y(0) = 1. \\ y &= \left(1 - \frac{15kt}{2\pi}\right)^{2/5} && \text{Take roots.} \end{aligned}$$

**Determination of** $k$. Let $V_0 = V(1)/10$ be the volume drained after $t_0 = 20$ seconds. Then $t_0$, $V_0$ and $k$ satisfy

$$\begin{aligned} V_0 &= V(1) - V(y(t_0)) && \text{Volume from height } y(t_0) \text{ to } y(0). \\ &= \frac{\pi}{9}\left(1 - y^3(t_0)\right) && \\ &= \frac{\pi}{9}\left(1 - \left(1 - \frac{15k}{2\pi}\,t_0\right)^{6/5}\right) && \text{Substitute (17).} \\ k &= \frac{2\pi}{15t_0}\left(1 - \left(1 - \frac{9V_0}{\pi}\right)^{5/6}\right) && \text{Solve for } k. \\ &= \frac{2\pi}{15t_0}\left(1 - 0.9^{5/6}\right) && \end{aligned}$$

**Drain times**. The volume is $V_h = \pi/18$ at time $t_1$ given by $\pi/18 = V(t_1)$ or in detail $\pi/18 = \pi y^3(t_1)/9$. This requirement simplifies to $y^3(t_1) = 1/2$. Then

| | |
|---|---|
| $\left(1 - \dfrac{15kt_1}{2\pi}\right)^{6/5} = \dfrac{1}{2}$ | Insert the formula for $y(t)$. |
| $1 - \dfrac{15kt_1}{2\pi} = \dfrac{1}{2^{5/6}}$ | Take the $5/6$ power of both sides. |
| $t_1 = \dfrac{2\pi}{15k}\left(1 - 2^{-5/6}\right)$ | Solve for $t_1$. |
| $= t_0 \dfrac{1 - 2^{-5/6}}{1 - 0.9^{5/6}}$ | Insert the formula for $k$. |
| $\approx 104.4$ | Half-tank drain time in seconds. |

The drain time $t_2$ for the full tank is not twice this answer but $t_2 \approx 2.28t_1$ or 237.9 seconds. The result is justified by solving for $t_2$ in the equation $y(t_2) = 0$, which gives $t_2 = \dfrac{2\pi}{15k} = \dfrac{t_1}{1 - 2^{-5/6}} = \dfrac{t_0}{1 - 0.9^{5/6}}$.

**40 Example (Stefan's Law)** An inmate sits unclothed in a room with skin temperature $33^\circ$ Celsius. The Celsius room temperature is given by $C(r) = 14 + 11r/20$ for $r$ in minutes. Assume in Stefan's law $k = \sigma A\mathcal{E} = 0.63504 \times 10^{-7}$. Find the number of joules lost through the skin in the first 20 minutes.

**Solution**: The theory implies that the answer is $Q(t_1)$ where $t_1 = (20)(60)$ is in seconds and $Q' = kT^4 - kT_0^4$. Seconds $(t)$ are converted to minutes $(r)$ by the equation $r = t/60$. Use $T = 33 + 273.15$ and $T_0(t) = C(t/60) + 273.15$, then

$$Q(t_1) = k\int_0^{t_1} (T^4 - (T_0(t))^4)dt \approx 110,103 \quad \text{joules.}$$

**41 Example (Tsunami)** Find a piecewise solution, which represents a Tsunami wave profile, similar to equation (9), on page 152. Graph the solution on $|x - x_0| \le 2$.

$$(y')^2 = 8y^2 - 4y^3, \quad x_0 = 1.$$

**Solution**: Equilibrium solutions $y = 0$ and $y = 2$ are found from the equation $8y^2 - 4y^3 = 0$, which has factored form $4y^2(2 - y) = 0$.

Non-equilibrium solutions with $y' \ge 0$ and $0 < y < 2$ satisfy the first order differential equation

$$y' = 2y\sqrt{2 - y}.$$

Consulting a computer algebra system gives the solution

$$y(x) = 2 - 2\tanh^2(\sqrt{2}(x - x_0)).$$

Treating $-y' = 2y\sqrt{2 - y}$ similarly results in exactly the same solution.

**Hand solution**. Start with the substitution $u = \sqrt{2 - y}$. Then $u^2 = 2 - y$ and $2uu' = -y' = -2yu = -2(2 - u^2)u$, giving the separable equation $u' = u^2 - 2$. Reformulate it as $u' = (u - a)(u + a)$ where $a = \sqrt{2}$. Normal partial fraction methods apply to find an implicit solution involving the inverse hyperbolic tangent. Some integral tables tabulate the integral involved, therefore partial fractions can be technically avoided. Solving for $u$ in the implicit equation gives the hyperbolic tangent solution $u = \sqrt{2}\tanh(\sqrt{2}(x - x_0))$. Then $y = 2 - u^2$ produces the answer reported above. The piecewise solution, which represents an ocean Tsunami wave, is given by

$$y(x) = \begin{cases} 2 & x \le 1, & \textbf{back-wave} \\ 2 - 2\tanh^2(\sqrt{2}(x-1)) & 1 < x < \infty. & \textbf{wave front} \end{cases}$$

The figure can be made by hand. A computer algebra graphic appears in Figure 12, with `maple` code as indicated.

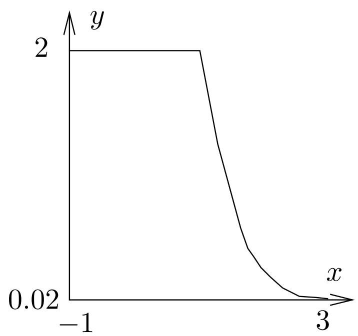



**Figure 12. Tsunami wave profile.** The back-wave is at height 2. The front wave has height given by the hyperbolic tangent term, which approaches zero as $x \to \infty$. The `maple` code:

```
g:=x->2-2*tanh(sqrt(2)*(x-1));
f:=x->piecewise(x<1,2,g(x));
plot(f,-1..3);
```

**42 Example (Gompertz Equation)** First, solve the Gompertz tumor equation, and then make (a) a phase line diagram and (b) a direction field.

$$y' = (8 - 2\ln y)y.$$

**Solution**: The only equilibrium solution computed from $G(y) \equiv (8 - 2\ln y)y = 0$ is $y = e^4 \approx 54.598$, because $y = 0$ is not in the domain of the right side of the differential equation.

Non-equilibrium solutions require integration of $1/G(y)$. Evaluation using a computer algebra system gives the implicit solution

$$-\frac{1}{2}\ln(8 - 2\ln(y)) = x + c.$$

Solving this equation for $y$ in terms of $x$ results in the explicit solution

$$y(x) = c_1 e^{-\frac{1}{2}e^{-2x}}, \quad c_1 = e^{4 - \frac{1}{2}e^{-2c}}.$$

The `maple` code for these two independent tasks appears below.

```
p:=int(1/((8-2*ln(y))*y),y);
solve(p=x+c,y);
```

The phase line diagram in Figure 13 requires the equilibrium $y = e^4$ and formulas $G(y) = (8 - 2\ln y)y$, $G'(y) = 8 - 2\ln y - 2$. Then $G'(e^4) = -2$ implies $G$ changes sign from positive to negative at $y = e^4$, making $y = e^4$ a stable sink or funnel.

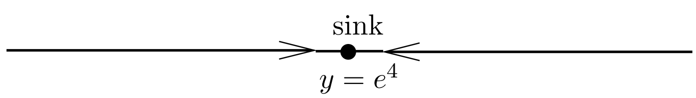



**Figure 13. Gompertz phase line diagram.**
The unique equilibrium at $y = e^4$ is a stable sink.

A computer-generated direction field appears in Figure 14, using the following `maple` code. Visible is the funnel structure at the equilibrium point.

```
de:=diff(y(x),x)=y(x)*(8-2*ln(y(x)));
with(DEtools):
DEplot(de,y(x),x=0..4,y=1..70);
```

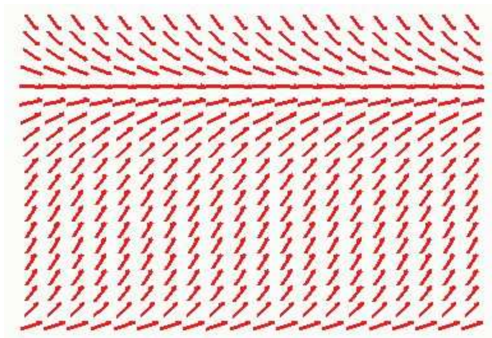


**Figure 14. A Gompertz direction field.**

## Details and Proofs

**Technical Details for (2):** The derivation of $\frac{d}{dt}(V(y(t))) = -k\sqrt{y(t)}$ uses Torricelli's speed formula $|v| = \sqrt{2gy(t)}$. The volume change in the tank for an orifice of cross-sectional area $a$ is $-av$. Therefore, $dV(y(t))/dt = -a\sqrt{2gy(t)}$. Succinctly, $dV(y(t))/dt = -k\sqrt{y(t)}$. This completes the verification.

**Technical Details for (3):** The equation $y'(t) = -k\dfrac{\sqrt{y(t)}}{A(y(t))}$ is equivalent to $A(y(t))\,y'(t) = -k\sqrt{y(t)}$. The equation $dV(y(t))/dt = V'(y(t))y'(t)$, obtained by the chain rule, definition $A(y) = V'(y)$, and equation (2) give result (3).

**Technical Details for (8):** To be verified is the Torricelli orifice equation $|v| = \sqrt{2gh}$ for the speed $|v|$ of a droplet falling from height $h$. Let's view the droplet as a point mass $m$ located at the droplet's centroid. The distance $x(t)$ from the droplet to the orifice satisfies a falling body model $mx''(t) = -mg$. The model has solution $x(t) = -gt^2/2 + x(0)$, because $x'(0) = 0$. The droplet arrives at the orifice in time $t$ given by $x(t) = 0$. Because $x(0) = h$, then $t = \sqrt{2h/g}$. The velocity $v$ at this time is $v = x'(t) = -gt = -\sqrt{2gh}$. A technically precise derivation can be done using kinetic and potential energy relations; some researchers prefer energy method derivations for Torricelli's law. Formulas for the orifice speed depend upon the shape and size of the orifice. For common drilled holes, the speed is a constant multiple $c\sqrt{2gh}$, where $0 < c < 1$.

**Technical Details for (11):** Assume $V = V_0 e^{\mu(t)}$ and $\mu(t) = a(1 - e^{-bt})/b$. Then $\mu' = ae^{-bt}$ and

| | |
|---|---|
| $V' = V_0\mu'(t)e^{\mu(t)}$ | Calculus rule $(e^u)' = u'e^u$. |
| $= \mu'(t)V$ | Use $V = V_0 e^{\mu(t)}$. |
| $= ae^{-bt}V$ | Use $\mu' = ae^{-bt}$. |

The equation $V' = ae^{-bt}V$ is a growth equation $y' = ky$ where $k$ decreases with time, causing the doubling time to increase. One biological explanation for the increase in the mean generation time of the tumor cells is aging of the reproducing cells, causing a slower dividing time. The correctness of this explanation is debatable.

Let $y = V/V_0$. Then

| | |
|---|---|
| $\frac{y'}{y} = \frac{V'}{V}$ | The factor $1/V_0$ cancels. |
| $= ae^{-bt}$ | Differential equation $V' = ae^{-bt}V$ applied. |
| $= a - b\mu(t)$ | Use $\mu(t) = a(1 - e^{-bt})/b$. |
| $= a - b\ \ln(V/V_0)$ | Take logs across $V/V_0 = e^{\mu(t)}$ to find $\mu(t)$. |
| $= a - b\ \ln y$ | Use $y = V/V_0$. |

Hence $y' = (a - b\ln y)y$. When $V \approx V_0$, then $y \approx 1$ and the growth rate $a - b\ln y$ is approximately $a$. Hence the model behaves like the exponential growth model $y' = ay$ when the tumor is small. The tumor grows subject to $a - b\ln y > 0$, which produces the volume restraint $\ln y = a/b$ or $V_{\max} = V_0 e^{a/b}$.

**Technical Details for (12):** Polar coordinates $r, \theta$ will be used. The geometry in the parabolic mirror Figure 10 shows that triangle $ABC$ is isosceles with angles $\alpha$, $\alpha$ and $\pi - 2\alpha$. Therefore, $\theta = 2\alpha$ is the angle made by segment $CA$ with the $x$-axis ($C$ is the origin $(0,0)$).

| | |
|---|---|
| $y = r\sin\theta$ | Polar coordinates. |
| $= 2r\sin\alpha\cos\alpha$ | Use $\theta = 2\alpha$ and $\sin 2x = 2\sin x\cos x$. |
| $= 2r\tan\alpha\cos^2\alpha$ | Identity $\tan x = \sin x/\cos x$ applied. |
| $= 2r\frac{dy}{dx}\cos^2\alpha$ | Use calculus relation $\tan\alpha = dy/dx$. |
| $= r\frac{dy}{dx}(1 + \cos 2\alpha)$ | Identity $2\cos^2 x - 1 = \cos 2x$ applied. |
| $= \frac{dy}{dx}(r + x)$ | Use $x = r\cos\theta$ and $2\alpha = \theta$. |

For $y > 0$, equation (12) can be solved as follows.

| | |
|---|---|
| $\frac{dx}{dy} = \frac{x}{y} + \sqrt{(x/y)^2 + 1}$ | Divide by $y$ on the right side of (12). |
| $y\frac{du}{dy} = \sqrt{u^2 + 1}$ | Substitute $u = x/y$ ($u$ cancels). |
| $\int \frac{du}{\sqrt{u^2 + 1}} = \int \frac{dy}{y}$ | Integrate the separated form. |
| $\sinh^{-1} u = \ln y$ | Integral tables. The integration constant is zero because $u(1) = 0$. |
| $\frac{x}{y} = \sinh(\ln y)$ | Let $u = x/y$ and apply $\sinh$ to both sides. |

$$= \frac{1}{2}\left(e^{\ln y} - e^{-\ln y}\right) \qquad \text{Definition } \sinh u = (e^u - e^u)/2.$$
$$= \frac{1}{2}\left(y - 1/y\right) \qquad \text{Identity } e^{\ln y} = y.$$

Clearing fractions in the last equality gives $2x + 1 = y^2$, a parabola of the form $X = Y^2$.

**Technical Details for (15):** Given polar coordinates $r$, $\theta$ and $\tan\alpha = dy/dx$, it will be shown that $r\,d\theta/dr = \tan(\alpha - \theta)$. Details require the formulas

$$\begin{aligned} x &= r\cos\theta, & \frac{dx}{dr} &= \cos\theta - r\frac{d\theta}{dr}\sin\theta, \\ y &= r\sin\theta, & \frac{dy}{dr} &= \sin\theta + r\frac{d\theta}{dr}\cos\theta. \end{aligned} \tag{18}$$

Then

$$\begin{aligned} \tan\alpha &= \frac{dy}{dx} && \text{Definition of derivative.} \\ &= \frac{dy/dr}{dx/dr} && \text{Chain rule.} \\ &= \frac{\sin\theta + r\frac{d\theta}{dr}\cos\theta}{\cos\theta - r\frac{d\theta}{dr}\sin\theta} && \text{Apply equation (18).} \\ &= \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta} && \text{Divide by } \cos\theta. \end{aligned}$$

Let $X = rd\theta/dr$ and cross-multiply to eliminate fractions. Then the preceding relation implies $(1 - X\tan\theta)\tan\alpha = \tan\theta + X$ and finally

$$\begin{aligned} r\frac{d\theta}{dr} &= X && \text{Definition of } X. \\ &= \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta} && \text{Solve for } X \text{ in } (1 - X\tan\theta)\tan\alpha = \tan\theta + X. \\ &= \tan(\alpha - \theta) && \text{Apply identity } \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}. \end{aligned}$$

Physicists and engineers often justify formula (15) referring to Figure 15. Such diagrams are indeed the initial intuition required to *guess* formulas like (15).

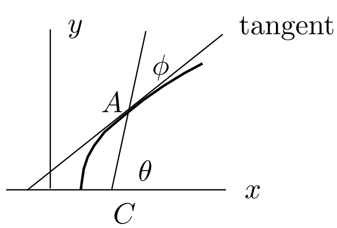



**Figure 15. Polar differential triangle.** Angle $\phi$ is the *signed angle* between the radial vector and the tangent line.

# Exercises 2.8

## Tank Draining.

**1.** A cylindrical tank 6 feet high with 6-foot diameter is filled with gasoline. In 15 seconds, 5 gallons drain out. Find the drain times for the next 20 gallons and the half-volume.

2. A cylindrical tank 4 feet high with 5-foot diameter is filled with gasoline. The half-volume drain time is 11 minutes. Find the drain time for the full volume.

3. A conical tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x$, $0 \le x \le 5$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

4. A conical tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x$, $0 \le x \le 5$. The units are in meters. Find the half-volume drain time for the tank, given the first 5 liters drain out in 10 seconds.

5. A spherical tank of diameter 12 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 20 seconds.

6. A spherical tank of diameter 9 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 3 seconds.

7. A hemispherical tank of diameter 16 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 25 seconds.

8. A hemispherical tank of diameter 10 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 4 seconds.

9. A parabolic tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x^2$, $0 \le x \le 2$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

10. A parabolic tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x^2$, $0 \le x \le 2$. The units are in meters. Find the half-volume drain time for the tank, given the first 4 liters drain out in 16 seconds.

**Torricelli's Law and Uniqueness.** It it known that Torricelli's law gives a differential equation for which Picard's existence-uniqueness theorem is inapplicable for initial data $y(0) = 0$.

11. Explain why Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ fails to satisfy the hypotheses in Picard's theorem. Cite all failed hypotheses.

12. Consider a typical Torricelli's law equation $y' = k\sqrt{y}$ with initial condition $y(0) = 0$. Argue physically that the depth $y(t)$ of the tank for $t < 0$ can be zero for an arbitrary duration of time $t$ near $t = 0$, even though $y(t)$ is not zero for all $t$.

13. Display infinitely many solutions $y(t)$ on $-5 \le t \le 5$ of Torricelli's equation $y' = k\sqrt{y}$ such that $y(t)$ is not identically zero but $y(t) = 0$ for $0 \le t \le 1$.

14. Does Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ have a solution $y(t)$ defined for $t \ge 0$? Is it unique? Apply Picard's theorem and Peano's theorem, if possible.

**Clepsydra: Water Clock Design.** A surface of revolution is used to make a container of height $h$ feet for a water clock. A curve $y = f(x)$ is revolved around the $y$-axis to make the container shape (e.g., $y = x$ makes a conical shape). Water drains out by gravity at $(0,0)$. The orifice has diameter $d$ inches. The water level in the tank must fall at a constant rate of $r$ inches per hour. Find $d$ and $f(x)$, given $h$ and $r$.

**15.** $h = 5$, $r = 4$

**16.** $h = 4$, $r = 4$

**17.** $h = 10$, $r = 7$

**18.** $h = 10$, $r = 8$

**19.** $h = 15$, $r = 10$

**20.** $h = 15$, $r = 8$

Stefan's Law. An unclothed prison inmate is handcuffed to a chair. The inmate's skin temperature is 33° Celsius. Given emissivity $\mathcal{E}$, skin area $A$ square meters and room temperature $T_0(r) = C(r/60)+273.15$, $r$ in seconds, find the number of Joules of heat lost by the inmate's skin after $t_0$ minutes. Use equation (5), page 150.

**21.** $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 10$, $C(t) = 24 + 7t/t_0$

**22.** $\mathcal{E} = 0.9$, $A = 1.7$, $t_0 = 12$, $C(t) = 21 + 10t/12$

**23.** $\mathcal{E} = 0.9$, $A = 1.4$, $t_0 = 10$, $C(t) = 15 + 15t/t_0$

**24.** $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 12$, $C(t) = 15 + 14t/t_0$

On the next two exercises, use a computer algebra system (CAS).

**25.** $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 15$, $C(t) = 15 + 15\sin\pi(t - t_0)/12$

**26.** $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 20$, $C(t) = 15 + 14\sin\pi(t - t_0)/12$

Tsunami Wave Shape. Plot the piecewise solution (9). See Figure 12.

**27.** $x_0 = 2$, $|x - x_0| \le 2$

**28.** $x_0 = 3$, $|x - x_0| \le 4$.

Tsunami Piecewise Solutions. Display a piecewise solution similar to (9). Produce a plot like Figure 12.

**29.** $x_0 = 2$, $|x - x_0| \le 4$, $(y')^2 = 16y^2 - 10y^3$.

**30.** $x_0 = 2$, $|x - x_0| \le 4$, $(y')^2 = 16y^2 - 12y^3$.

**31.** $x_0 = 3$, $|x - x_0| \le 4$, $(y')^2 = 8y^2 - 2y^3$.

**32.** $x_0 = 4$, $|x - x_0| \le 4$, $(y')^2 = 16y^2 - 4y^3$.

Tsunami Wavefront. Find non-equilibrium solutions for the given differential equation.

**33.** $(y')^2 = 16y^2 - 10y^3$.

**34.** $(y')^2 = 16y^2 - 12y^3$.

**35.** $(y')^2 = 8y^2 - 2y^3$.

**36.** $(y')^2 = 16y^2 - 4y^3$.

Gompertz Tumor Equation. Solve the Gompertz tumor equation $y' = (a - b\ln y)y$. Make a phase line diagram.

**37.** $a = 1$, $b = 1$

**38.** $a = 1$, $b = 2$

**39.** $a = -1$, $b = 1$

**40.** $a = -1$, $b = 2$

**41.** $a = 4$, $b = 1$

**42.** $a = 5$, $b = 1$