

**Math 2250 Extra Credit Maple Project 5: Linear Algebra
S2015**

Extra credit Maple lab 5 has four problems: L5-1, L5-2, L5-3, L5-4. Examples of the `maple` coding required appears in five examples at the end of this document.

References: Code in `maple` appears in `2250mapleL5-S2015.txt` at URL <http://www.math.utah.edu/~gustafso/>. This document: `2250mapleL5-S2015.pdf`.

Problem L5-1. (Matrix Algebra)

Define $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$. Create a worksheet in `maple` which states this problem in text, then defines the four objects. The worksheet should contain text, `maple` code and displays. Continue with this worksheet to answer (1)–(7) below. Submit problem L5.1 as a worksheet printed on 8.5 by 11 inch paper. See Example 1 for `maple` commands.

- (1) Compute AB and BA . Are they the same?
- (2) Compute $A + B$ and $B + A$. Are they the same?
- (3) Let $C = A + B$. Compare C^2 to $A^2 + 2AB + B^2$. Explain why they are different.
- (4) Compute transposes $C_1 = (AB)^T$, $C_2 = A^T$ and $C_3 = B^T$. Find a matrix equation for C_1 in terms of C_2 and C_3 . Verify the equation.
- (5) Solve for \mathbf{X} in $B\mathbf{X} = \mathbf{v}$ by `maple` commands `linalg[rref]`, `linalg[linsolve]`, then by inversion $\mathbf{X} = A^{-1}\mathbf{b}$.
- (6) Solve $A\mathbf{Y} = \mathbf{v}$ for \mathbf{Y} . Do an answer check using `linalg[linsolve]`.
- (7) Solve $A\mathbf{Z} = \mathbf{w}$. Explain your answer using the three possibilities for a linear system. Discuss the possible `maple` reports for (1) no solution case, (2) unique solution, (3) infinitely many solutions.

Problem L5-2. (Independent Columns)

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 1 & 2 & -3 & -1 & -9 \end{pmatrix}.$$

Find independent vectors which have the same span as the columns of A using the following methods.

Method 1. Find the pivot columns of A . See Example 2.

Method 2. The `maple` command `linalg[colspace](A)`.

The first method is equivalent to finding a largest set of independent vectors from the list of 5 vectors formed from the columns of A . The answer is a basis of 2 vectors. The span of these 2 vectors equals the span of the 5 column vectors of A . The second method finds another basis of 2 vectors, which is generally different, but equivalent in the sense described in the next part.

Problem L5-3. (Equivalent Bases)

Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Verify that the two bases $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ are **equivalent**. This means that each vector in \mathcal{B}_1 is a linear combination of the vectors in \mathcal{B}_2 , and conversely, that each vector in \mathcal{B}_2 is a linear combination of the vectors in \mathcal{B}_1 . Briefly,

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}.$$

Staple this page on top of the `maple` work sheets.

Problem L5-4. (Matrix Equations)

Let $A = \begin{pmatrix} 8 & 10 & 3 \\ -3 & -5 & -3 \\ -4 & -4 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Let P denote a 3×3 matrix. Define $\lambda_1 = 1$, $\lambda_2 = -2$ and $\lambda_3 = 5$.

Assume the following result:

Lemma 1. The equality $AP = PT$ holds if and only if the columns \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 of P satisfy $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$. [proved after Example 5]

(a) Determine three specific columns for P such that $\det(P) \neq 0$ and $AP = PT$. These columns contain only numbers – no symbols allowed! Infinitely many answers are possible. See Example 5 for the maple method that determines a column of P .

(b) After reporting the three columns, check the answer by computing $AP - PT$ (it should be zero) and $\det(P)$ (it should be nonzero).

Example 1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$. Create a maple work sheet. Define and display matrix A and vector \mathbf{b} . Then compute

- (1) The inverse of A .
- (2) The augmented matrix $C = \mathbf{aug}(A, \mathbf{b})$.
- (3) The reduced row echelon form $R = \mathbf{rref}(C)$.
- (4) The column \mathbf{X} of R which solves $A\mathbf{X} = \mathbf{b}$.
- (5) The matrix A^3 .
- (6) The transpose of A .
- (7) The matrix $A - 3A^2$.
- (8) The solution \mathbf{X} of $A\mathbf{X} = \mathbf{b}$ by two methods different than (4).
- (9) Find a matrix F such that $F\mathbf{x} = \mathbf{b}$ has no solution. Explain why `linsolve` prints nothing.
- (10) Compute $A^T A$, $(A^T A)^{-1}$, $A^{-1}(A^{-1})^T$.

Solution: A lab instructor or classmate can help you to create a blank work sheet in `maple`, enter code and print the work sheet. The code to be entered into the work sheet appears below.

```
A:=Matrix([[1,2,3],[2,-1,1],[3,0,-1]]);
b:=Vector([9,8,3]);
print("(1)"); A^(-1);
print("(2)"); C:=<A|b>;
print("(3)"); R:=linalg[rref](C);
print("(3) Alternate");
R:=LinearAlgebra[ReducedRowEchelonForm](C);
print("(4)"); R[1..3,4]; # Select col=4 from 3x4 matrix R
print("(4) Alternate"); X:=LinearAlgebra[Column](R,4);
print("(5)"); A^3;
print("(6)"); A^+;
print("(7)"); A-3*A^2;
print("(8)"); X:=A^(-1).b;X:=LinearAlgebra[LinearSolve](A,b);
# (9): linsolve prints nothing: a signal equation "0=3".
print("(9)"); F:=Matrix([[1,2,3],[2,-1,1],[0,0,0]]);linalg[linsolve](F,b);
<F|b>; # Visual explanation of the signal equation
print("(10)"); A^+ . A; (A^+.A)^(-1); A^(-1) . (A^(-1))^+;
```

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$.

- (1) Find a basis for the column space of A . This means: find a largest list of independent columns of A .
- (2) Find a basis for the row space of A .
- (3) Find a basis for the nullspace of A . This is the list of vector partials $\partial_{t_1}\mathbf{x}, \partial_{t_2}\mathbf{x}, \dots$ applied to the general solution \mathbf{x} of $A\mathbf{x} = \mathbf{0}$, which is obtained from the *last frame algorithm*.
- (4) Find $\mathbf{rank}(A)$ and $\mathbf{nullity}(A)$. They are the number of lead variables and the number of free variables for the problem $A\mathbf{x} = \mathbf{0}$, respectively.
- (5) Find the dimensions of the nullspace, row space and column space of A .

Solution: The theory applied: *The columns of B corresponding to the leading ones in $\mathbf{rref}(B)$ are independent and form a basis for the column space of B .* These columns are called the **pivot columns** of B . The meaning is

$$\mathbf{span}\{\text{all columns of } B\} = \mathbf{span}\{\text{pivot columns of } B\}.$$

A list of vectors is called a **basis** provided it is **independent** and **spans**.

Results for the row space of A are obtained by replacing B by the transpose of A . In particular, the row space of A is spanned by the pivot columns of $B = A^T$.

The `maple` code which applies is

```
A:=Matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
print("(1)"); C:=linalg[rref](A); # leading ones in columns 1,2,4
           BASIScolumnspace=A[1..4,1],A[1..4,2],A[1..4,4];
print("(2)"); F:=linalg[rref](A^+); # leading ones in columns 1,2,3
           BASISrowspace=row(A,1),row(A,2),row(A,3);
print("(3)"); linalg[nullspace](A); linalg[linsolve](A,Vector([0,0,0,0]));

print("(4)"); RANK=linalg[rank](A);
           NULLITY=linalg[coldim](A)-linalg[rank](A);

print("(5)"); DIMnullspace=linalg[coldim](A)-linalg[rank](A);
           DIMrowspace=linalg[rank](A);
           DIMcolumnspace=linalg[rank](A);
```

Example 3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$. Verify that the following column space bases of A are equivalent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 17 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -9 \end{pmatrix}.$$

Solution: We will use this result:

Lemma 2. Bases $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are equivalent bases if and only if the augmented matrices $F = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, $G = \mathbf{aug}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and $H = \mathbf{aug}(F, G)$ satisfy the rank condition $\mathbf{rank}(F) = \mathbf{rank}(G) = \mathbf{rank}(H) = 3$.

The proof appears below.

The `maple` code which applies is

```

A:=Matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
v1:=Vector([1,2,3,4]); v2:=Vector([1,3,5,3]); v3:=Vector([2,1,1,2]);
w1:=Vector([1,0,0,-3]); w2:=Vector([0,1,0,17]); w3:=Vector([0,0,1,-9]);
F:=<v1|v2|v3>;
G:=<w1|w2|w3>;
H:=<F|G>;
linalg[rank](F); linalg[rank](G); linalg[rank](H);

```

Example 4. The two bases in the example were discovered from the `maple` code

```

A:=Matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
linalg[rref](A); # pivot cols 1,2,4
v1:=A[1..4,1]; v2:=A[1..4,2]; v3:=A[1..4,4];# Name the columns
linalg[rref](A^+); # pivot cols 1,2,3
w1:=A[1,1..5]; w2:=A[2,1..5]; w3:=A[3,1..5];# Name the rows

```

Proof of Lemma 2.

Proof: Let's justify part of the test, showing only half the proof: $\mathbf{rank}(F) = \mathbf{rank}(G) = \mathbf{rank}(H) = n$ implies the bases are equivalent.

The equation $\mathbf{rref}(F) = EF$ holds for E a product of elementary matrices. Then $\mathbf{rref}(H) = EH$ has to have n leading ones, because of F in the first n columns, and the remaining rows of $\mathbf{rref}(H)$ are zero, because $\mathbf{rank}(H) = n$. Therefore, the first n columns of $H = \mathbf{aug}(F, G)$ are the pivot columns of H . The non-pivots of H are just the columns of G , and by the pivot theory, they are linear combinations of the pivot columns, namely, the columns of F . We can multiply H by a permutation matrix P which effectively swaps F and G . Already, HP has the n independent columns of F , so its rank is at least n . But its other columns are linear combinations of these columns, so the rank is exactly n . Now we argue by symmetry that the columns of F are linear combinations of the columns of G , using HP instead of H .

The first half of the proof is complete. The other half is left to the reader.

Example 5. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$. Solve the equation $A\mathbf{x} = -3\mathbf{x}$ for \mathbf{x} .

Solution. Let $\lambda = -3$. The idea is to write the equation $A\mathbf{x} = \lambda\mathbf{x}$ as a homogeneous problem $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

The trick is to move $\lambda\mathbf{x}$ from the RHS to the LHS of the equation, then re-write $\lambda\mathbf{x}$ as $\lambda I\mathbf{x}$, where I is the identity matrix. Then \mathbf{x} is a common factor, and the matrix equation can be written as $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Define $B = A - \lambda I$. The homogeneous equation $B\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$. It has a nonzero solution \mathbf{x} if and only if there are infinitely many solutions, in which case the solutions are found by a frame sequence to $\mathbf{rref}(B)$. The `maple` details appear below. The basis vectors for $B\mathbf{x} = \mathbf{0}$ are obtained in the usual way, by taking partial derivatives on the general solution with respect to the symbols t_1, t_2, \dots . In this case, there is just one basis vector

$$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

```

A:=Matrix([[1,2,3],[2,-1,1],[3,0,0]]);
B:=A-(-3)*<1,0,0|0,1,0|0,0,1>;
linalg[linsolve](B,Vector([0,0,0]));
# ans: Vector([-2*t,1*t,2*t]), maple replaces t by _t_1
# Basis == partial on t == Vector([-2,1,2])

```

Proof of Lemma 1. Define $r_1 = 1, r_2 = -2, r_3 = 5$. Assume $AP = PT, P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $T = \mathbf{diag}(r_1, r_2, r_3)$. The definition of matrix multiplication implies that $AP = \mathbf{aug}(A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3)$ and $PT = \mathbf{aug}(r_1\mathbf{v}_1, r_2\mathbf{v}_2, r_3\mathbf{v}_3)$. Then $AP = PT$ holds if and only if the columns of the two matrices match, which is equivalent to the three equations $A\mathbf{v}_1 = r_1\mathbf{v}_1, A\mathbf{v}_2 = r_2\mathbf{v}_2, A\mathbf{v}_3 = r_3\mathbf{v}_3$. The proof is complete.

End of Maple Lab 5 Linear Algebra.