Sample Quiz 7

Sample Quiz 7, Problem 1. Independence

The Problem. In the parts below, cite which tests apply to decide on independence or dependence. Choose one test and show complete details.

(a) Vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) Vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

(c) Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are data packages constructed from the equations $y = x, y = x^2, y = x^4$ on $(-\infty, \infty)$.

(d) Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are data packages constructed from the equations $y = 10 + x, y = 10 - x, y = x^5$ on $(-\infty, \infty)$.

Basic Test. To show three vectors are independent, form the system of equations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0},$$

then solve for c_1, c_2, c_3. If the only solution possible is $c_1 = c_2 = c_3 = 0$, then the vectors are independent.

Linear Combination Test. A list of vectors is independent if and only if each vector in the list is not a linear combination of the remaining vectors, and each vector in the list is not zero.

Subset Test. Any nonvoid subset of an independent set is independent.

The basic test leads to three quick independence tests for column vectors. All tests use the augmented matrix A of the vectors, which for 3 vectors is $A = < \vec{v}_1 | \vec{v}_2 | \vec{v}_3 >$.

Rank Test. The vectors are independent if and only if $\mathbf{rank}(A)$ equals the number of vectors.

Determinant Test. Assume A is square. The vectors are independent if and only if the determinant of A is nonzero.

Pivot Test. The vectors are independent if and only if all columns of A are pivot columns. The maximum number of independent columns equals the rank, which is the number of pivot columns.

The basic test leads to independence tests for functions. Throughout, f_1, f_2, f_3 are given functions to be tested for independence on an interval $a < x < b$. The tests extend in the obvious way to 2 or more functions.

Sampling Test. Invent samples x_1, x_2, x_3 from $a < x < b$ and form the sample matrix $S = \begin{pmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{pmatrix}$. If $|S| \neq 0$, then the functions are independent.

Wronskian Test. Form the Wronskian matrix $W(x) = \begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{pmatrix}$.

If $|W(x)| \neq 0$ for some x in $a < x < b$, then the functions are independent.

Euler Solution Atom Test. Any list of distinct Euler Solution Atoms is independent. Atoms are $1, \cos(bx), \sin(bx)$ with $b > 0$, or these three multiplied by e^{ax} with $a \neq 0$, or any of the preceding multiplied by a positive integer power of x, that is, by $x, x^2, x^3, \ldots$.

Sample Quiz 7, Problem 2. Subspaces

The Problem. Decide if each of the following sets is a subspace. Cite the theorem(s) below that apply and supply details.

1. Vector space $V = \mathcal{R}^3$. Set S is defined by relations $x_1 + x_2 = 4x_3, x_1 + 2x_3 = 0$.

2. Vector space $V = \mathcal{R}^3$. Set S is defined by relations $x_1 = x_2, x_2 + x_3 = 5$.

3. Vector space V of all polynomials $p(x)$. Set S is defined by relation $p'(1) = 0$.

4. Vector space V of all 2×2 matrices A. Set S is defined by $A \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

5. Vector space $V = \mathcal{R}^3$. Set S is all vectors with irrational entries.

6. Vector space $V = \mathcal{R}^3$. Set S is all vectors perpendicular to $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

7. Vector space $V = \mathcal{R}^3$. Set S is all vectors with $x_1 x_2 x_3 = 0$.

8. Vector space V of all continuous functions. Set S is all linear combinations of 1 and e^{-x}.

Subspace Criterion. A subset S of a vector space V is a **subspace** provided the following three items are satisfied.

(a) The zero vector $\vec{0}$ is in S.

(b) If $\vec{v}_1, \vec{v}_2$ are in S, then their sum $\vec{v}_1 + \vec{v}_2$ is in S.

(c) If $\vec{v}$ is in S and c is a scalar, then $c\vec{v}$ is in S.

Kernel Theorem. A subset S of vector space $\mathcal{R}^n$ defined by a system of linear homogeneous algebraic equations is a subspace. Such equations can always be written as $A\vec{x} = \vec{0}$. The solution set of $A\vec{x} = \vec{0}$ is called the **kernel** of matrix A, or alternatively, the **null space** of matrix A.

Span Theorem. The span of a set of vectors in a vector space V is a subspace of V. This means the set of all linear combinations of a list of vectors is a subspace of V.

Not a Subspace Theorem. A subset S of a vector space V fails to be a subspace if one of the following holds.

(a) The zero vector $\vec{0}$ is not in S.

(b) Two vectors $\vec{v}_1, \vec{v}_2$ in S can be defined, for which their sum $\vec{v}_1 + \vec{v}_2$ is not in S.

(c) A vector $\vec{v}$ in S can be defined for which $-\vec{v}$ is not in S.

A non-homogeneous system of linear equations $A\vec{x} = \vec{b}$ with $\vec{b} \neq \vec{0}$ can never define a subspace S, because $\vec{x} = \vec{0}$ is not a solution of the system. For instance, $x + y + z = 1$ in $\mathcal{R}^3$ cannot define a subspace.

Sample Quiz 7, Extra Credit Problem 1. Archeology and the Dot Product.

Archeologist Sir Flinders Petrie collected and analyzed pottery fragments from 900 Egyptian graves. He deduced from the data an historical ordering of the 900 sites. Petrie's ideas will be illustrated for 4 sites and 3 pottery types. The matrix rows below represent sites 1, 2, 3, 4 and the matrix columns represent pottery types 1, 2, 3. A matrix entry is 1 if the site has that pottery type and 0 if not. This is an **incidence matrix**.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Petrie Matrix. It is an incidence matrix in which the ones in each column appear together, like the matirx above.

Counting Pottery Types. The dot product of row 2 and row 3 is

$$\begin{pmatrix} 0,1,1 \end{pmatrix} \cdot \begin{pmatrix} 1,0,1 \end{pmatrix} = 0*1+1*0+1*1 = 1$$

which means sites 2 and 3 have one pottery type in common. **Please pause on this arithmetic**, until you agree that the products $0*1, 1*0, 1*1$ add to the number of pottery types in common.

Sites with pottery in common are expected to be historically close in time. Because pottery types evolve, old types cease production when newly created pottery types begin production, which gives meaning to the clustered ones in the columns of A. The ordering obeys for $n = 4$ sites the **Petrie Property**:

> *The number of pottery types site k has in common with sites 1 to n forms a sequence of numbers $a_1, a_2, \ldots, a_n$ which increases until a_k and then decreases, more precisely, $a_1 \le a_2 \le \cdots \le a_k$ and $a_k \ge a_{k-1} \ge \cdots \ge a_n$.*

To illustrate, site 2 generates $a_1, a_2, a_3, a_4 = 1, 2, 1, 1$ which increases until a_2, corresponding to row 2, then the sequence values decrease. Terms *decrease* and *increase* allow equality between elements.

Petrie matrix columns can be decoded as historical events. The 3rd column $[0, 1, 1, 0]$ means no type 3 pottery fragments were found at sites 1, 4 but fragments were found at sites 2, 3. The zeros place sites 1, 4 outside the time period of pottery type 3 manufacturing.

The Problem. By computing many dot products of pairs of rows of A, display computations which show that the **Petrie Property** holds.

In detail. The object is to compute 4 sequences of 4 numbers. Sequence 1 is the list $[a_1, a_2, a_3, a_4]$ of dot product answers for Row 1 against Rows 1,2,3,4. This computation is repeated for Row 2, Row 3 and Row 4. After a list is found, then the Petrie Property is checked.

References. Edwards-Penney section 3.6. Mathematics Magazine (1984) article by Alan Shucat: Matrix and Network Models in Archeology, which also discusses the connection between this problem and the traveling saleman problem. W. S. Robinson discussed the method for chronologically ordering archeological deposits, in the April 1951 issue of American Antiquity.

Sample Quiz 7, Extra Credit Problem 2. Archeology, the Transpose and Robinson's Matrix.

Definition. Given an incidence matrix A, then $R = AA^T$ is called the **Robinson Matrix** for A.

Illustration. Given the Petrie matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then the corresponding **Robinson matrix** is

$$R = AA^T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Kendall discovered that A is a Petrie matrix if and only if matrix $R = AA^T$ has the **Robinson Property**:

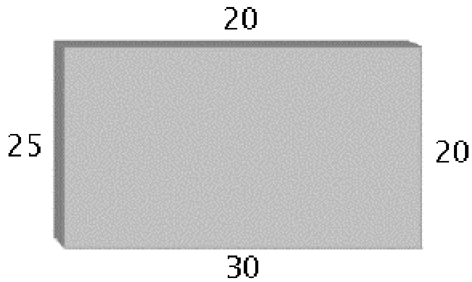
> *Consider any diagonal entry x of A, and let ordered list L_1 be constructed from the column entries upwards from x. Let list L_2 be constructed from the row entries to the right of x. Then both lists L_1, L_2 are non-increasing.*

The property can be checked visually in seconds, for the above displayed matrix R. For instance, $x = 2$ in row 2 has $L_1 = [1]$ and $L_2 = [1, 0]$, both non-increasing (decreasing with equality allowed).

The Problem. Each entry of the matrix product $R = AA^T$ is a dot product, specifically a row of A times a column of A^T. Explain why R contains the dot product answers of **Problem 1**. Your written explanation might help you to understand Kendall's discovery.

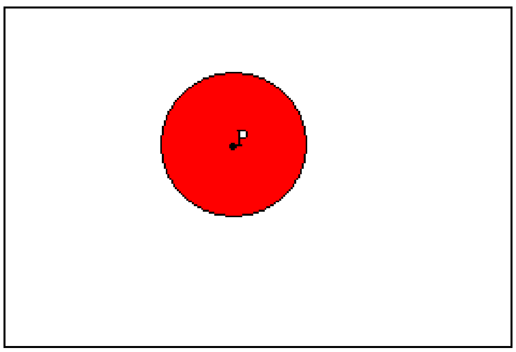
References. Edwards-Penney Sections 3.6 (transpose). David Kendall's 1969 work, Incidence matrices, interval graphs and seriation in archeology. W. S. Robinson, the method for chronologically ordering archeological deposits, in the April 1951 issue of American Antiquity.

Sample Quiz 7, Extra Credit Problem 3. **Heat Transfer** and the Mean Value Property. Consider the cross section of a long rectangular dam on a river, represented in the figure.



The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

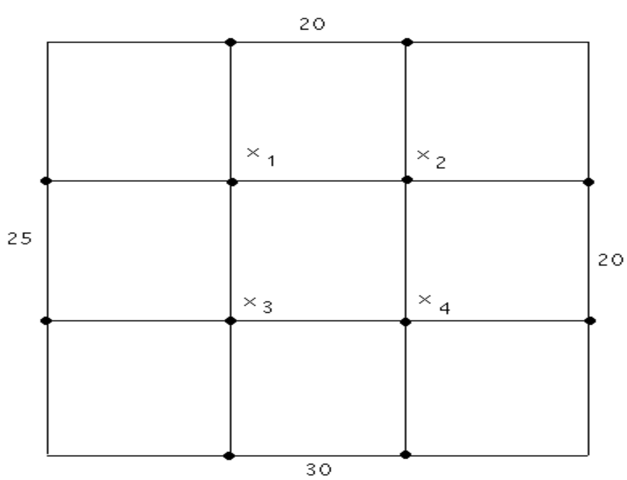
An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.



The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle contained in the plate with center P, then the temperature at P is the average value of the temperature function over C.

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C. We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4, then use the property to obtain four equations.



Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$ by any method.

References. EPH Chapters 12, 13, on heat transfer. Used in Partial Differential Equations 3150.