Sample Quiz 7, Solutions

problem 1. Independence

(a) Column vector tests are: Basic test, Linear combination test, Rank Test, Determinant Test, pivot test. All of These apply to (a). Because $\begin{pmatrix}1\\-2\end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix}1\\1\end{pmatrix}$, Then Test 2 applies, and They are independent. Remarks below apply to The other possible Tests.

→ The augmented matrix $A = \begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}$ has rank 2 and $|A|=3$, so The rank and determinant Tests apply. Because rank = 2, Then There are 2 pivot cols, and The pivot Test applies.

(b) The easiest Test is That $\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\-1\end{pmatrix}$, so They are dependent. Other Tests are The Basic Test, Rank Test, Determinant Test and The pivot test. Let $A = \begin{pmatrix}1 & 1 & 2\\-1 & 1 & 0\\0 & -1 & -1\end{pmatrix}$ be The augmented matrix. Then combo(1,2,1) produces frame 2 = $\begin{pmatrix}1 & 1 & 2\\0 & 2 & 2\\0 & -1 & -1\end{pmatrix}$ and Combo(3,2,2) gives $\begin{pmatrix}1 & 1 & 2\\0 & 0 & 0\\0 & -1 & -1\end{pmatrix}$. Then rank(A) = 2, $|A|=0$, The number of pivots = 2. The Tests imply Dependent.

Remark: we never use The basic test, if an easier one is available.

(c) The vectors form a subset of vectors made from equations $y=1$, $y=x$, $y=x^2$, $y=x^3$, $y=x^4$. Because subsets of independent sets are independent, and atom list $1, x, x^2, x^3, x^4$ is independent Then The given vectors are independent. Other tests That apply are The Sampling test with samples $x=1,2,3$ and The Wronskian Test at $x=1$.

(d) Because $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathrm{Span}(1, x, x^5)$, and independent set $1, x, x^2, x^3, x^4, x^5$ (Euler Solution atoms) contains Subset $1, x, x^5$, Then The vectors are independent. Sampling & Wronskian Tests also apply.

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problem 2. Subspaces

① Linear algebraic homogeneous equations $x_1 + x_2 - 4x_3 = 0$, $x_1 + 2x_3 = 0$. The Kernel Theorem applies. Always, The Subspace Criterion applies, so we omit details for That.

② Linear algebraic non-homogeneous equations $x_1 - x_2 = 0$, $x_2 + x_3 = 5$. Because $x_1 = x_2 = x_3 = 0$ is not a solution, Then The Not A Subspace Theorem applies. Not a Subspace.

③ If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, Then $p'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$ and $p'(1) = a_1 + 2a_2 + \cdots + n a_n = 0$. This is a homogeneous linear algebraic equation, so by The Kernel Theorem it is a subspace.

④ The Subspace Criterion applies. If $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, Then $A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If A_1, A_2 work in The equation, Then $(A_1 + A_2)\begin{pmatrix} 1 \\ -2 \end{pmatrix} = A_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + A_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $A_1 + A_2$ works. If A works in The equation and c is a constant, Then $(cA)\begin{pmatrix} 1 \\ -2 \end{pmatrix} = c\left(A\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = c\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ So also cA works. Then S is a subspace by The Subspace Criterion. None of The other results apply.

⑤ Not a Subspace, because $\vec{0}$ has all entries rational. See The Not a Subspace Test.

⑥ Set S is The matrix equation $A\vec{x} = \vec{0}$, where $A = (1 \ \ 1 \ \ 0) = 1\times 3$ matrix. The Kernel Theorem says it is a subspace. The Subspace Criterion also applies.

⑦ Not a Subspace, because $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ satisfy $x_1 x_2 x_3 = 0$, but Their sum $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ satisfies $x_1 x_2 x_3 = 1 \neq 0$. The Not a Subspace Theorem applies. Nothing else directly decides.

⑧ $S = \text{span}(1, e^{x})$ = subspace by The Span Theorem. Only The Subspace Criterion gives an alternate test, but it duplicates The proof of The Span Theorem.

Sample Quiz 7 Solutions

Extra credit.
problem 1.
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Row 1 · Row 1 = $(0\ 1\ 0)\cdot(0\ 1\ 0) = 1$
Row 1 · Row 2 = $(0\ 1\ 0)\cdot(0\ 1\ 1) = 1$
Row 1 · Row 3 = $(0\ 1\ 0)\cdot(1\ 0\ 1) = 0$
Row 1 · Row 3 = $(0\ 1\ 0)\cdot(1\ 0\ 0) = 0$

[1 1 0 0]
Decreases from a_1
petrie property holds

Row 2 · Row 1 = $(0\ 1\ 1)\cdot(0\ 1\ 0) = 1$
Row 2 · Row 2 = $(0\ 1\ 1)\cdot(0\ 1\ 1) = 2$
Row 2 · Row 3 = $(0\ 1\ 1)\cdot(1\ 0\ 1) = 1$
Row 2 · Row 4 = $(0\ 1\ 1)\cdot(1\ 0\ 0) = 0$

increases
To here
Then decreases
[1 2 1 0]
a_2

Row 3 · Row 1 = $(1\ 0\ 1)\cdot(0\ 1\ 0) = 0$
Row 3 · Row 2 = $(1\ 0\ 1)\cdot(0\ 1\ 1) = 1$
Row 3 · Row 3 = $(1\ 0\ 1)\cdot(1\ 0\ 1) = 2$
Row 3 · Row 4 = $(1\ 0\ 1)\cdot(1\ 0\ 0) = 1$

[0 1 2 1]
a_3
Increases to a_3
Then decreases.

Row 4 · Row 1 = $(1\ 0\ 0)\cdot(0\ 1\ 0) = 0$
Row 4 · Row 2 = $(1\ 0\ 0)\cdot(0\ 1\ 1) = 0$
Row 4 · Row 3 = $(1\ 0\ 0)\cdot(1\ 0\ 1) = 1$
Row 4 · Row 4 = $(1\ 0\ 0)\cdot(1\ 0\ 0) = 1$

[0 0 1 1]
a_4
increases to a_4,
Then no more data

Sample Quiz 7 Solutions

Extra Credit

Problem 2. Explain why $R = AA^T$ contains the dot product answers of Problem 1.

Each answer in problem 1 is a dot product of 2 rows. A column of A^T is a row of A.

Multiplying a row of A against a column of A^T gives the same answer as a dot product of 2 rows of A.

The 4 blocks of answers in problem 1 compute the 4 rows of $R = AA^T$.

<u>Remark</u>. The Robinson property is easy to check by machine. It's purpose is to take a non-Petrie matrix C and swap rows of C until it becomes a Petrie Matrix A. Swaps are elementary matrices

$$E_1, E_2, \ldots, E_K$$

So

$$E_K \cdots E_2 E_1 C = A$$

and then the Robinson matrices at each stage are $(EC_p)(EC_p)^T = EC_pC_p^TE^T$, the next Robinson Matrix C_{p+1} is obtained by simultaneous row and column swaps.

Sample Quiz 7 Solutions

Extra Credit

Problem 3. write the equations as

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 45 \\ 40 \\ 55 \\ 50 \end{pmatrix}$$

The vector $\vec{x}$ is the vector of equilibrium temperatures, located at the grid points of the diagram.

we do swap, combo, mult on the augmented matrix

$$\left(\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 45 \\ -1 & 4 & 0 & -1 & 40 \\ -1 & 0 & 4 & -1 & 55 \\ 0 & -1 & -1 & 4 & 50 \end{array}\right)$$

or alternatively compute the inverse of the coefficient matrix A:

$$A^{-1} = \begin{pmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{pmatrix} \cdot \frac{1}{24}$$

The solution of $A\vec{x} = \vec{b}$ is always $\vec{x} = A^{-1}\vec{b}$, so

$$\vec{x} = \frac{1}{24}\begin{pmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{pmatrix} \begin{pmatrix} 45 \\ 40 \\ 55 \\ 50 \end{pmatrix} = \begin{pmatrix} 23.125 \\ 21.875 \\ 25.625 \\ 24.375 \end{pmatrix} = \begin{pmatrix} 185 \\ 175 \\ 205 \\ 195 \end{pmatrix} \cdot \frac{1}{8}$$

most of this can be done with technology. As the grid size goes up, so does the difficulty of writing $A, \vec{b}$.