Chapter 10

Phase Plane Methods

Contents

Studied here are planar autonomous systems of differential equations. The topics:

Planar Autonomous Systems: Phase Portraits, Stability.

Planar Constant Linear Systems: Classification of isolated equilibria, Phase portraits.

Planar Almost Linear Systems: Phase portraits, Nonlinear classifications of equilibria.

Biological Models: Predator-prey models, Competition models, Survival of one species, Co-existence, Alligators, doomsday and extinction.

Mechanical Models: Nonlinear spring-mass system, Soft and hard springs, Energy conservation, Phase plane and scenes.

10.1 Planar Autonomous Systems

A set of two scalar differential equations of the form

$$(1)\qquad \begin{aligned} x'(t) &= f(x(t), y(t)), \\ y'(t) &= g(x(t), y(t)). \end{aligned}$$

is called a **planar autonomous system**. The term **autonomous** means **self-governing**, justified by the absence of the time variable t in the functions $f(x, y)$, $g(x, y)$.

To obtain the vector form, let $\vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\vec{F}(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$ and write (1) as the first order vector-matrix system

$$(2)\qquad \frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t)).$$

It is assumed that f, g are continuously differentiable in some region $\mathcal{D}$ in the xy-plane. This assumption makes $\vec{F}$ continuously differentiable in $\mathcal{D}$ and guarantees that Picard's existence-uniqueness theorem for initial value problems applies to the initial value problem $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$, $\vec{u}(0) = \vec{u}_0$. Accordingly, to each $\vec{u}_0 = (x_0, y_0)$ in $\mathcal{D}$ there corresponds a unique solution $\vec{u}(t) = (x(t), y(t))$, represented as a planar curve in the xy-plane, which passes through $\vec{u}_0$ at $t = 0$.

Such a planar curve is called a **trajectory** or **orbit** of the system and its parameter interval is some maximal interval of existence $T_1 < t < T_2$, where T_1 and T_2 might be infinite. A graphic of trajectories drawn as parametric curves in the xy-plane is called a **phase portrait** and the xy-plane in which it is drawn is called the **phase plane**.

Trajectories Don't Cross

Autonomy of the planar system plus uniqueness of initial value problems implies that trajectories $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ cannot touch or cross. Hand-drawn phase portraits are accordingly limited: *you cannot draw a solution trajectory that touches another solution curve*!

Theorem 1 (Identical Trajectories)
Assume that Picard's existence-uniqueness theorem applies to initial value problems in $\mathcal{D}$ for the planar system

$$\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t)), \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two trajectories of the system. If times t_1, t_2 exist such that

$$(3)\qquad x_1(t_1) = x_2(t_2), \quad y_1(t_1) = y_2(t_2),$$

then for the value $c = t_1 - t_2$ the equations $x_1(t+c) = x_2(t)$ and $y_1(t+c) = y_2(t)$ are valid for all allowed values of t. This means that the two trajectories are on one and the same planar curve, or in the contrapositive, two different trajectories cannot touch or cross in the phase plane.

Proof: Define $x(t) = x_1(t+c)$, $y(t) = y_1(t+c)$. By the chain rule, $(x(t), y(t))$ is a solution of the planar system, because $x'(t) = x_1'(t+c) = f(x_1(t+c), y_1(t+c)) = f(x(t), y(t))$, and similarly for the second differential equation. Further, (3) implies $x(t_2) = x_2(t_2)$ and $y(t_2) = y_2(t_2)$, therefore Picard's uniqueness theorem implies that $x(t) = x_2(t)$ and $y(t) = y_2(t)$ for all allowed values of t. The proof is complete.

Equilibria

A trajectory that reduces to a point, or a constant solution $x(t) = x_0$, $y(t) = y_0$, is called an **equilibrium solution**. The equilibrium solutions or **equilibria** are found by solving the nonlinear equations

$$f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0.$$

Each such (x_0, y_0) in $\mathcal{D}$ is a trajectory whose graphic in the phase plane is a single point, called an **equilibrium point**. In applied literature, it may be called a **critical point**, **stationary point** or **rest point**. Theorem 1 has the following geometrical interpretation.

> Assuming uniqueness, no other trajectory $(x(t), y(t))$ in the phase plane can touch an equilibrium point (x_0, y_0).

Equilibria (x_0, y_0) are often found from linear equations

$$ax_0 + by_0 = e, \quad cx_0 + dy_0 = f,$$

which are solved by linear algebra methods. They constitute an important subclass of algebraic equations which can be solved symbolically. In this special case, symbolic solutions exist for the equilibria.

It is interesting to report that in a practical sense the equilibria may be reported incorrectly, due to the limitations of computer software, even in the case when exact symbolic solutions are available. An example is $x' = x + y$, $y' = \epsilon y - \epsilon$ for small $\epsilon > 0$. The root of the problem is translation of ϵ to a machine constant, which is zero for small enough ϵ. The result is that computer software detects infinitely many equilibria when in fact there is exactly one equilibrium point. This example suggests that symbolic computation be used by default.

Practical Methods for Computing Equilibria

There exists no supporting theory to find equilibria for all choices of F and G. However, there is a rich library of special methods for solving nonlinear algebraic equations, including numerical methods based on celebrated univariate methods, such as **Newton's method** and the **bisection method**.

Computer algebra systems like `maple`, `maxima` and `mathematica` offer convenient codes to solve the equations, when possible, including symbolic solutions. Applied mathematics depends on the dynamically expanding library of special methods, which grows due to new mathematical discoveries. See the exercises for examples.

Population Biology

Planar autonomous systems have been applied to two-species populations like two species of trout, who compete for food from the same supply, and foxes and rabbits, who compete in a predator-prey situation.

Certain equilibria are significant, because they represent the population sizes for **cohabitation**. A point in the phase space that is not an equilibrium point corresponds to population sizes that cannot coexist, they must change with time. Some equilibria are consequently **observable** or **average** population sizes while non-equilibria correspond to snapshot population sizes that are subject to flux. Biologists expect population sizes of such two-species competition models to undergo change until they reach approximately the observable values, on the average.

Rabbit-Fox System

This example is a **predator-prey** system, in which the expected observable population sizes are averages, about which the actual populations size oscillate about, periodically over time. Certain equilibria for these systems represent **ideal cohabitation**. Biological experiments suggest that initial population sizes close to the equilibrium values cause populations to stay near the initial sizes, even though the populations oscillate periodically. Observations by field biologists of large population variations seem to verify that individual populations oscillate periodically around the ideal cohabitation sizes.

A typical planar system for predator-prey dynamics of $x(t)$ rabbits and

$y(t)$ foxes is the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{200}x(40-y),\\ \frac{dy}{dt} &= \frac{1}{100}y(x-50).\end{aligned}$$

Time variable t is in months. The equilibria are $(0,0),(50,40)$. With initial populations $x(0)=60$ rabbits and $y(0)=30$ foxes, both x' and y' are positive near $t=0$, which implies the populations initially increase in size.

After time, the signs of x' and y' are alternately positive and negative, which reflects the oscillating behavior of the populations about the ideal equilibrium values $x=50$, $y=40$. The period of oscillation is about 20 months. This predator-prey model predicts coexistence with average populations of 50 rabbits and 40 foxes.

Trout System

Consider a population of two species of trout who compete for the same food supply. A typical autonomous planar system for the species x and y is

$$\begin{aligned}\frac{dx}{dt} &= x(-2x-y+180),\\ \frac{dy}{dt} &= y(-x-2y+120).\end{aligned}$$

Equilibria. The equilibrium solutions for the trout system are

$$(0,0),\quad (90,0),\quad (0,60),\quad (80,20).$$

Only nonnegative population sizes are physically significant. Units for the population sizes might be in hundreds or thousands of fish. The equilibrium $(0,0)$ corresponds to **extinction** of both species, while $(0,60)$ and $(90,0)$ correspond to the unusual situation of extinction for one species. The last equilibrium $(80,20)$ corresponds to **co-existence** of the two trout species with observable population sizes of 80 and 20.

Phase Portraits

A graphic which contains some equilibria and typical trajectories of a planar autonomous system (1) is called a **phase portrait**.

While graphing equilibria is not a challenge, graphing typical trajectories, also called **orbits**, seems to imply that we are going to solve the differential system. This is not the case. Approximations will be used that do not require solution of the differential system.

Equilibria	Plot in the xy-plane all equilibria of (1). See Figure 3.
Window	Select an x-range and a y-range for the graph window which includes all significant equilibria (Figure 3).
Grid	Plot a uniform grid of N grid points ($N \approx 50$ for hand work) within the graph window, to populate the graphical white space (Figure 4). The isocline method might also be used to select grid points.
Field	Draw at each grid point a short tangent vector, a **replacement curve** for a solution curve through a grid point on a small time interval (Figure 5).
Orbits	Draw additional threaded trajectories on long time intervals into the remaining white space of the graphic (Figure 6). This is guesswork, based upon tangents to threaded trajectories matching nearby field tangents drawn in the previous step. See Figures 1 and 2 for details.

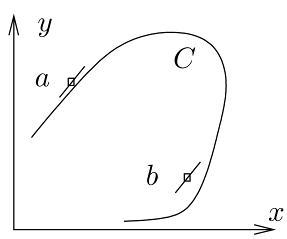

Figure 1. Badly threaded orbit.

Threaded solution curve C correctly matches its tangent to the tangent at nearby grid point a, but it fails to match at grid point b.

Why does a threaded solution curve tangent $\vec{T}_1$ have to match a tangent $\vec{T}_2$ at a nearby grid point (see Figure 2)? A tangent vector is given by $\vec{T} = \frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$. Then $\vec{T}_1 = \vec{F}(\vec{u}_1)$, $\vec{T}_2 = \vec{F}(\vec{u}_2)$. However, $\vec{u}_1 \approx \vec{u}_2$ in the graphic, hence by continuity of $\vec{F}$ it follows that $\vec{F}(\vec{u}_1) \approx \vec{F}(\vec{u}_2)$, which implies $\vec{T}_1 \approx \vec{T}_2$.

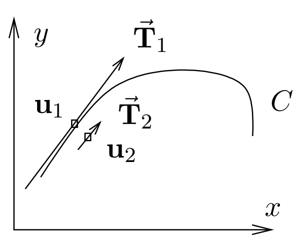

Figure 2. Tangent matching.

Threaded solution curve C matches its tangent $\vec{T}_1$ at $\vec{u}_1$ to direction field tangent $\vec{T}_2$ at nearby grid point $\vec{u}_2$.

It is important to emphasize that solution curves starting at a grid point are defined for a small t-interval about $t = 0$, and therefore their graphics extend on both sides of the grid point. We intend to shorten these curves until they appear to be straight line segments, graphically atop the tangent line, to pixel resolution. Adding an arrowhead pointing in the tangent vector direction is usual. After all this construction, *the shaft of the arrow is graphically atop a short solution curve segment.* In fact, if 50 grid points were used, then 50 short solution curve segments have already been entered onto the graphic! Threaded orbits are added to show what happens to solutions that are plotted on longer and longer t-intervals.

Phase Portrait Illustration

The method outlined above will be applied to the illustration

$$(4) \qquad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

The equilibria are $(1, -1)$ and $(-1, 1)$. The graph window is selected as $|x| \leq 2$, $|y| \leq 2$, in order to include both equilibria. The uniform grid will be 11×11, although for hand work 5×5 is normal. Tangents at the grid points are short line segments which do not touch each another – they are graphically the same as short solution curves.

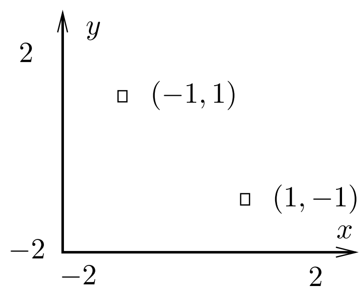

Figure 3. Equilibria $(1, -1)$, $(-1, 1)$ with Invented Graph Window.

The equilibria (x, y) are calculated from equations $0 = x + y$, $0 = 1 - x^2$. The graph window $|x| \leq 2$, $|y| \leq 2$ is invented initially, then updated until Figure 5 reveals sufficiently rich field details.

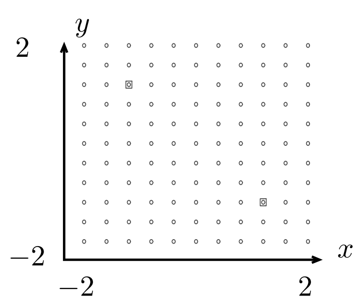

Figure 4. Equilibria $(1, -1)$, $(-1, 1)$ and Invented 11×11 Uniform Grid.

The equilibria (squares) happen to cover up two grid points (circles). The invented size 11×11 is to fill the white space in the graphic.

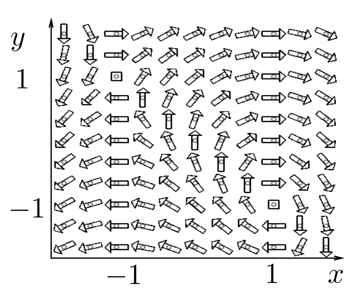

Figure 5. Equilibria, Uniform Grid and Direction Field.

An arrow shaft at a grid point represents a solution curve over a small time interval. Threaded solution curves on long time intervals have tangents matching nearby arrow shaft directions.

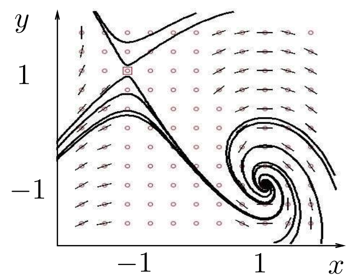

Figure 6. Initial Phase Portrait. Equilibria $(1, -1)$, $(-1, 1)$ and 11×11 uniform grid with threaded solution curves. Arrow shafts included from some direction field arrows.

Threaded solution curve tangents are to match nearby direction field arrow shafts. See Figures 1 and 2 for how to match tangents.

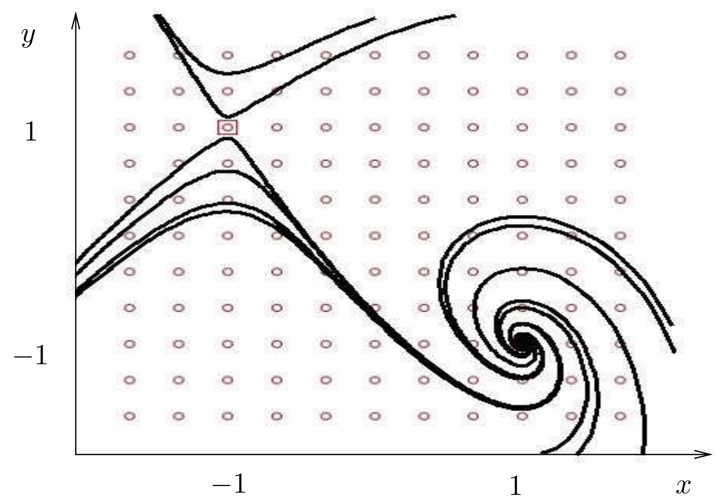

Figure 7. Final Phase portrait.
Shown are some threaded solution curves and an 11×11 grid. The direction field has been removed for clarity. Threaded solution curves do not actually cross, even though graphical resolution might suggest otherwise.

Phase Plot by Computer

Illustrated here is how to make a phase plot like Figure 8 or Figure 9, *infra*, with computer algebra system `maple`, for the system of differential equations

$$(5) \qquad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

Before the computer work begins, the differential equation is defined and the equilibria are computed. Defaults supplied by `maple` allow an initial phase portrait to be plotted, from which the graph window is invented.

Phase plot tools can simplify initial plot production. To illustrate, `maple` task **Phase Portrait** has this interface:

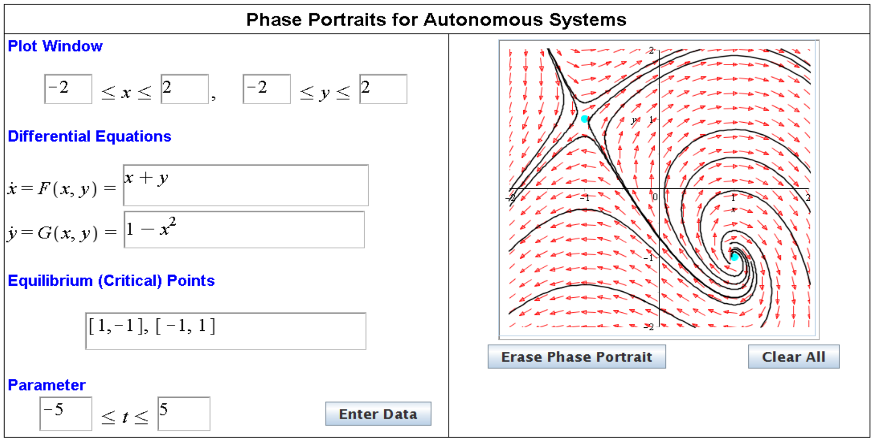

Figure 8. `PhasePortrait` **task in computer algebra system** `Maple` **for equations (5).**

Minimal input requires two differential equations, equilibria, a graph window and time interval for threaded curves. Clicking on the graphic produces threaded solution curves.

The Phase Portrait Task is unlikely to be able to produce a final, production figure. Other tools are normally used afterwards, to make the final figure.

The initial plot code:

```
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:Times:=t=-20..20:
DEtools[DEplot]([des],[x(t),y(t)],Times,wind);
```

The initial plot suggests which initial conditions near the equilibria should be selected in order to create typical orbits on the graphic. The final code with initial data and options:

```
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:Times:=t=-20..20:
opts:=stepsize=0.05,dirgrid=[13,13],
axes=none,thickness=3,arrows=small:
ics:=[[x(0)=-1,y(0)=1.1],[x(0)=-1,y(0)=1.5],
[x(0)=-1,y(0)=.9],[x(0)=-1,y(0)=.6],[x(0)=-1,y(0)=.3],
[x(0)=1,y(0)=-0.9],[x(0)=1,y(0)=-0.6],[x(0)=1,y(0)=-0.6],
[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=-1.6],[x(0)=1,y(0)=-1.3],
[x(0)=1,y(0)=-1.1]]:
DEtools[DEplot]([des],[x(t),y(t)],Times,wind,ics,opts);
```

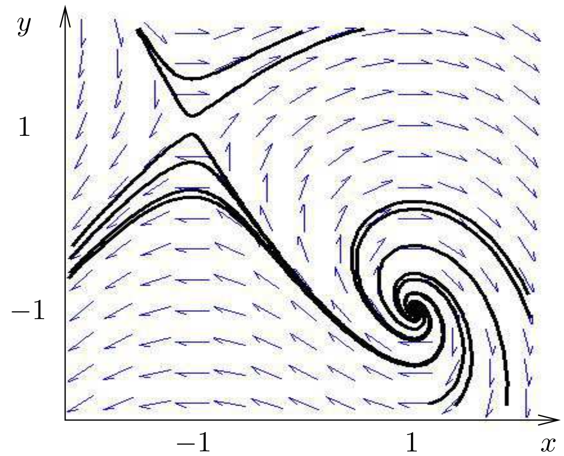

Figure 9. Phase Portrait for (5).

The graphic shows typical solution curves and a direction field. The graphic was produced in `maple` using a 13×13 grid.

Stability

Consider an autonomous system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ with $\vec{F}$ continuously differentiable in a region $\mathcal{D}$ in the plane.

Stable equilibrium. An equilibrium point $\vec{u}_0$ in $\mathcal{D}$ is said to be **stable** provided for each $\epsilon > 0$ there corresponds $\delta > 0$ such that

(a) given $\vec{u}(0)$ in $\mathcal{D}$ with $\|\vec{u}(0) - \vec{u}_0\| < \delta$, then the solution $\vec{u}(t)$ exists on $0 \le t < \infty$ and

(b) $\|\vec{u}(t) - \vec{u}_0\| < \epsilon$ for $0 \le t < \infty$.

Unstable equilibrium. The equilibrium point $\vec{u}_0$ is called **unstable** provided it is **not stable**, meaning at least one of **(a)** or **(b)** fails.

Asymptotically stable equilibrium. The equilibrium point $\vec{u}_0$ is said to be **asymptotically stable** provided **(a)** and **(b)** hold (it is **stable**), and additionally

(c) $\lim_{t\to\infty} \|\vec{u}(t) - \vec{u}_0\| = 0$ for $\|\vec{u}(0) - \vec{u}_0\| < \delta$.

Applied accounts of stability tend to emphasize item **(b)**. Careful application of stability theory requires attention to **(a)**, which is the question of extension of solutions of initial value problems to the half-axis.

Basic extension theory for solutions of autonomous equations says that **(a)** will be satisfied provided **(b)** holds for those values of t for which $\vec{u}(t)$ is already defined. Stability verifications in mathematical and applied literature often implicitly use extension theory, in order to present details compactly. The reader is advised to adopt the same predisposition as researchers, who assume the reader to be equally clever as they.

Physical stability. In the model $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$, physical stability addresses changes in $\vec{F}$ as well as changes in $\vec{u}(0)$. The meaning is this: physical parameters of the model, e.g., the mass $m > 0$, damping constant $c > 0$ and Hooke's constant $k > 0$ in a damped spring-mass system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{c}{m}\,y - \frac{k}{m}\,x, \end{aligned}$$

may undergo small changes without significantly affecting the solution.

In physical stability, stable equilibria correspond to **physically observed** data whereas other solutions correspond to **transient observations** that disappear over time.

A typical instance is the trout system

$$(6) \qquad \begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

Physically observed data in the trout system (6) corresponds to the **carrying capacity**, represented by the **stable equilibrium** point $(80, 20)$, whereas transient observations are snapshot population sizes that are subject to change over time. The strange extinction equilibria $(90, 0)$ and $(0, 60)$ are **unstable equilibria**, which disagrees with intuition about zero births for less than two individuals, but agrees with graphical representations of the trout system in Figure 10. Changing f for a trout system adjusts the physical constants which describe the birth and death rates, whereas changing $\vec{u}(0)$ alters the initial population sizes of the two trout species.

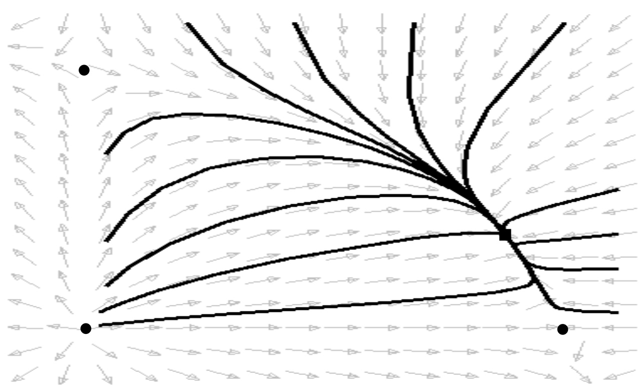

Figure 10. Phase Portrait for Trout System (6).

Shown are typical solution curves and a direction field. Equilibrium $(80, 20)$ is asymptotically stable (a square). Equilibria $(0, 0)$, $(90, 0)$, $(0, 60)$ are unstable (circles).

Direction Fields by Computer

Direction fields are produced by `Maple` with its `DEplot` tool, or with the graphical task `PhasePortrait`. Basic code that produces a direction field can be written with minimal outside support. The ideas discussed below apply to other programming languages, such as `Maxima, Mathematica, Ruby, Python` and Microsoft developer languages.

The `Maple` code below considers the system

$$x' = F_1(x, y), \quad y' = F_2(x, y)$$

with example $x' = F_1 = x + y, y' = F_2 = 1 - x^2$, which was treated above. Used are Maple libraries `plots` and `plottools`.

The `plottools` function `rectangle` requires two arguments ul, lr, which are the upper left (ul) and lower right (lr) vertices of the rectangle.

The `plottools` function `arrow` requires five arguments P, Q, sw, aw, af: the two points P, Q which define the arrow shaft and direction, plus the shaft width sw, arrowhead width aw and arrowhead length fraction af (fraction of the shaft length).

The two functions `rectangle, arrow` plot a polygon from its vertices. Function `rectangle` computes four vertices and function `arrow` computes seven vertices. Function `plots[display]` plots the vertices.

```
F1:=(x,y)->evalf(x+y):F2:=(x,y)->evalf(1-x^2): # Define system
a:=-2:b:=2:c:=-2:d:=2:n:=11:m:=11:             # Window and Grid

# 2D phase plane direction field with uniform nxm grid.
# Tangent length is 9/10 the grid box width W0.

H:=evalf((b-a)/(n+1)):K:=evalf((d-c)/(m+1)):W0:=min(H,K):
X:=t->a+H*(t):Y:=t->c+K*(t):P:=[]:
for i from 1 to n do
 for j from 1 to m do
  x:=X(i):y:=Y(j):M1:=F1(x,y): M2:=F2(x,y):
  if (M1 =0 and M2 =0) then # no tangent, make a box
   h:=W0/5:V:=plottools[rectangle]([x-h,y+h],[x+h,y-h]):
  else
   h:=evalf(((1/2)*9*W0/10)/sqrt(M1^2+M2^2)):
   p1:=x-h*M1:p2:=y-h*M2:q1:=x+h*M1:q2:=y+h*M2:
   V:=plottools[arrow]([p1,p2],[q1,q2],0.2*W0,0.5*W0,1/4):
  fi:
  if (P = []) then P:=V: else P:=P,V: fi:
od:od:
plots[display](P);
```

Exercises 10.1

Autonomous Planar Systems.

Consider

$$(7) \quad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

1. **(Vector-Matrix Form)** System (7) can be written in vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

2. **(Picard's Theorem)** Picard's vector existence-uniqueness theorem applies to system (7) with initial data $x(0) = x_0$, $y(0) = y_0$. Show the details.

Trajectories Don't Cross.

3. **(Theorem 1 Details)** Compute $\frac{dy}{dt} = g(x_1(t+c), y_1(t+c))$, then show that $y'(t) = g(x(t), y(t))$ in the proof of Theorem 1.

4. **(Orbits Can Cross)** The example

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3y^{2/3}$$

has infinitely many orbits crossing at $x = y = 0$. Exhibit two distinct orbits which cross at $x = y = 0$. Does this example contradict Theorem 1?

Equilibria.

A point (x_0, y_0) is called an **equilibrium** provided $x(t) = x_0$, $y(t) = y_0$ is a solution of the dynamical system.

5. Justify that $(1, -1), (-1, 1)$ are the only equilibria for the system $x' = x + y$, $y' = 1 - x^2$.

6. Display the details which justify that $(0,0), (90,0), (0,60), (80,20)$ are all equilibria for the system $x'(t) = x(-2x - y + 180)$, $y'(t) = y(-x - 2y + 120)$.

Practical Methods for Computing Equilibria.

7. **(Murray System)** The biological system

$$x' = x(6 - 2x - y), y' = y(4 - x - y)$$

has equilibria $(0,0)$, $(3,0)$, $(0,4)$, $(2,2)$. Justify the four answers.

8. **(Nullclines)** Curves along which either $x' = 0$ or $y' = 0$ are called **nullclines**. The biological system

$$x' = x(6 - 2x - y), y' = y(4 - x - y)$$

has nullclines $x = 0$, $y = 0$, $6 - 2x - y = 0$, $4 - x - y = 0$. Justify the four answers.

9. **(Nullclines by Computer)** Produce a graphical display of the nullclines of the Murray System above. Maple code to produce a nullcline plot is as follows

```
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=0..130,y=0..80;
plots[contourplot](eqns,wind,
  contours=[0]);
```

10. **(Isoclines by Computer)** Level curves $f(x, y) = c$ are called **isoclines**.

`Maple` will plot level curves $f(x, y) = -2$, $f(x, y) = 0$, $f(x, y) = 2$ using the nullcline code above, with replacement `contours=[-2,0,2]`. Produce an isocline plot for the Murray System above with these same contours.

11. **(Implicit Plot)** Equilibria can be found graphically by an implicit plot. `Maple` code:

```
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=0..130,y=0..80;
plots[implicitplot](eqns,wind);
```

Produce the implicit plot. Is it the same as the nullcline plot?

Rabbit-Fox System.

12. (Predator-Prey) Consider a rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

Argue why extinction of the rabbits ($x = 0$) implies extinction of the foxes ($y = 0$).

13. (Predator-Prey) The rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(40-y), \\ y' &= \frac{1}{100}y(x-40), \end{aligned}$$

has extinction of the foxes ($y = 0$) implying Malthusian population explosion of the rabbits ($\lim_{t=\infty} x(t) = \infty$). Explain.

Trout System.

Consider

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

14. (Carrying Capacity) Show details for calculation of the carrying capacities $x = 80$, $y = 20$.

15. (Stability) Equilibrium point $x = 80$, $y = 20$ is stable. Explain this statement using geometry from Figure 10 and the definition of stability.

Phase Portraits.

Consider

$$\begin{aligned} x'(t) &= x(t)+y(t), \\ y'(t) &= 1-x^2(t). \end{aligned}$$

16. (Equilibria) Solve for x, y in the system

$$\begin{aligned} 0 &= x+y, \\ 0 &= 1-x^2, \end{aligned}$$

for equilibria $(1, -1)$, $(-1, 1)$.

17. (Graph Window) Explain why $-2 \le x \le 2$, $-2 \le y \le 2$ is a suitable window.

18. (Grid Points) Draw a 5×5 grid on the graph window $|x| \le 2, |y| \le 2$. Label the equilibria.

19. (Direction Field) Draw direction field arrows on the 5×5 grid of the previous exercise. They coincide with the tangent direction $\vec{v} = x'\vec{\imath} + y'\vec{\jmath} = (x+y)\vec{\imath} + (1-x^2)\vec{\jmath}$, where (x, y) is the grid point. The arrows may not touch.

20. (Threaded Orbits) On the direction field of the previous exercise, draw orbits (*threaded solution curves*), using the rules:

1. Orbits don't cross.
2. Orbits pass direction field arrows with nearly matching tangent.

Phase Plot by Computer.

Use a computer algebra system or a numerical workbench to produce phase portraits for the given dynamical system. A graph window should contain all equilibria.

21. (Rabbit-Fox System I)

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

22. (Rabbit-Fox System II)

$$\begin{aligned} x' &= \frac{1}{100}x(50-y), \\ y' &= \frac{1}{200}y(x-40). \end{aligned}$$

23. (Trout System I)

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

24. (Trout System II)

$$\begin{aligned} x'(t) &= x(-2x-y+200), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

Stability Inequalities.

The signs of $x'(t)$ and/or $y'(t)$ can predict stability or instability. Consider an equilibrium point (x_0, y_0) and all solutions $x(t), y(t)$ satisfying for H small the inequalities

$$|x(0) - x_0| \leq H, \quad |y(0) - y_0| \leq H.$$

25. (Instability: Repeller) Prove that $x'(t) > 0$ and $y'(t) > 0$ for all small $H > 0$ implies instability at x_0, y_0.

26. (Stability: Attractor) Prove that $x'(t) < 0$ and $y'(t) < 0$ for all small $H > 0$ implies stability at x_0, y_0.

27. (Instability in x) Prove that $x'(t) > 0$ for all small $H > 0$ implies instability at x_0, y_0.

28. (Instability in y) Prove that $y'(t) > 0$ for all small $H > 0$ implies instability at x_0, y_0.

Geometric Stability.

29. (Attractor) Imagine a dust particle in a fluid draining down a funnel, whose trace is a space curve. Project the space curve onto the plane orthogonal to the centerline of the funnel. Is this planar orbit stable at centerline position in the sense of the definition?

30. (Repeller) Imagine a paint droplet from a paint spray can, which traces a space curve. Project the space curve onto the plane orthogonal to the spray orifice direction. Is this planar orbit stable at centerline position in the sense of the definition?

Algebraic Stability.

31. (Rabbit–Fox Stability) Provide algebraic details for stability of equilibrium $x = 40$, $y = 30$ for the system

$$\begin{aligned} x' &= \frac{1}{200} x(30 - y), \\ y' &= \frac{1}{100} y(x - 40). \end{aligned}$$

32. (Rabbit–Fox Instability) Provide algebraic details for instability of equilibrium $x = 0$, $y = 0$ for the system

$$\begin{aligned} x' &= \frac{1}{100} x(50 - y), \\ y' &= \frac{1}{200} y(x - 40). \end{aligned}$$

10.2 Planar Constant Linear Systems

A **constant linear** planar system is a set of two scalar differential equations of the form

$$(1)\qquad \begin{aligned} x'(t) &= ax(t) + by(t)), \\ y'(t) &= cx(t) + dy(t)), \end{aligned}$$

where a, b, c and d are constants. In matrix form,

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Solutions drawn in phase portraits don't cross, because of Picard's theorem. The system is autonomous. The origin is always an equilibrium solution. There can be infinitely many equilibria, found by solving $A\vec{u} = \vec{0}$ for the constant vector $\vec{u}$, when A is not invertible.

Formula. System (1) can be solved by a formula which parallels the theorem for second order constant coefficient equations $Ay'' + By' + Cy = 0$. The reader is invited to learn Putzer's spectral method, page 753, which is used to derive the formulas. For now, we will accept the formulas displayed in the next theorem. Putzer's result depends only on the Cayley-Hamilton theorem, which says that a matrix A satisfies the characteristic equation $|A - \lambda I| = 0$ under substitution $\lambda = A$.

Theorem 2 (Planar Constant Linear System: Putzer's Formula)
Consider the real planar system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution $\vec{u}(t)$ is given by the formula

$$\vec{u}(t) = \Phi(t)\vec{u}(0)$$

where the 2×2 real invertible matrix $\Phi(t)$ is defined as follows.

Real $\lambda_1 \neq \lambda_2$	$\Phi(t) = e^{\lambda_1 t} I + \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} (A - \lambda_1 I).$
Real $\lambda_1 = \lambda_2$	$\Phi(t) = e^{\lambda_1 t} I + te^{\lambda_1 t} (A - \lambda_1 I).$
Complex $\lambda_1 = \overline{\lambda}_2$, $\lambda_1 = a + bi$, $b > 0$	$\Phi(t) = e^{at} \left(\cos(bt) I + (A - aI)\dfrac{\sin(bt)}{b}\right).$

Continuity and Redundancy

The formulas are continuous in the sense that limiting $\lambda_1 \to \lambda_2$ in the first formula or $b \to 0$ in the last formula produces the middle formula for real equal roots. The first formula is also valid for complex conjugate roots λ_1, $\lambda_2 = \overline{\lambda}_1$ and it reduces to the third when $\lambda_1 = a + ib$, therefore the third formula is technically redundant, but nevertheless useful, because it contains no complex numbers.

Recommended: Memorize the first formula, derive the other two.

About the Newton Quotient. The Newton quotient $\frac{g(x)-g(x_0)}{x-x_0}$ in the first formula of the theorem uses $g(x) = e^{xt}$, $x = \lambda_2$, $x_0 = \lambda_1$, $x - x_0 = \lambda_2 - \lambda_1$. Calculus defines $g'(x_0)$ as the Newton quotient limit as $x \to x_0$.

Illustrations

Typical cases are represented by the following 2×2 matrices A. The two roots λ_1, λ_2 of the characteristic equation must fall into one of the three possibilities: real distinct, real equal or complex conjugate.

$\lambda_1 = 5$, $\lambda_2 = 2$ — Real distinct roots.

$$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix} \qquad \vec{u}(t) = \left(e^{5t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{2t} - e^{5t}}{2-5} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix} \right) \vec{u}(0).$$

$\lambda_1 = \lambda_2 = 3$ — Real equal roots.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \qquad \vec{u}(t) = e^{3t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} \vec{u}(0).$$

$\lambda_1 = \overline{\lambda_2} = 2 + 3i$ — Complex conjugate roots.

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \qquad \vec{u}(t) = e^{2t} \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix} \vec{u}(0).$$

Isolated Equilibria

An autonomous system is said to have an **isolated equilibrium** at $\vec{u} = \vec{u}_0$ provided $\vec{u}_0$ is the only constant solution of the system in $|\vec{u} - \vec{u}_0| < r$, for $r > 0$ sufficiently small.

Theorem 3 (Isolated Equilibrium)
The following are equivalent for a constant planar system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$:

1. The system has an isolated equilibrium at $\vec{u} = \vec{0}$.
2. $\det(A) \neq 0$.
3. The roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ satisfy $\lambda_1\lambda_2 \neq 0$.

Proof: The expansion $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$ shows that $\det(A) = \lambda_1\lambda_2$. Hence **2** $\equiv$ **3**. We prove now **1** $\equiv$ **2**. If $\det(A) = 0$, then $A\vec{u} = \vec{0}$ has infinitely many solutions $\vec{u}$ on a line through $\vec{0}$, therefore $\vec{u} = \vec{0}$ is not an isolated equilibrium. If $\det(A) \neq 0$, then $A\vec{u} = \vec{0}$ has exactly one solution $\vec{u} = \vec{0}$, so the system has an isolated equilibrium at $\vec{u} = \vec{0}$.

Classification of Isolated Equilibria

For linear equations

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t),$$

we explain the phase portrait classifications

spiral, center, saddle, node

near the isolated equilibrium point $\vec{u} = \vec{0}$, and how to detect them when they occur. Below, λ_1, λ_2 are the roots of $\det(A - \lambda I) = 0$.

The reader is directed to Figures 13–12 for illustrations of the classifications. See also duplicate Figures 18–16, which are organized by geometry.

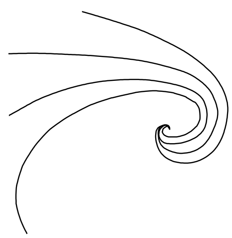

Figure 11. Spiral

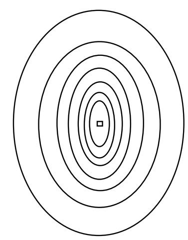

Figure 12. Center

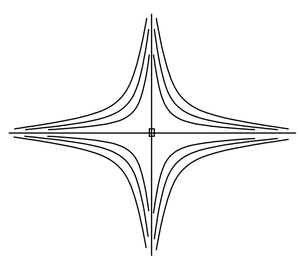

Figure 13. Saddle

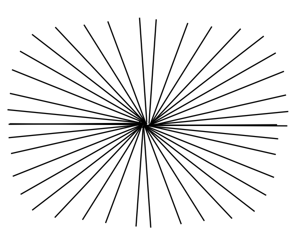

Figure 14. Proper node

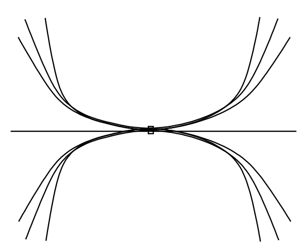

Figure 15. Improper node

Spiral $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a \neq 0$, $b > 0$.

A **spiral** has solution formula

$$\begin{aligned}\vec{u}(t) &= e^{at}\cos(bt)\,\vec{c}_1 + e^{at}\sin(bt)\,\vec{c}_2,\\ \vec{c}_1 &= \vec{u}(0), \quad \vec{c}_2 = \frac{A - aI}{b}\,\vec{u}(0).\end{aligned}$$

All solutions are bounded harmonic oscillations of natural frequency b times an exponential amplitude which grows if $a > 0$ and decays if $a < 0$. An orbit in the phase plane **spirals out** if $a > 0$ and **spirals in** if $a < 0$.

Center $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a = 0$, $b > 0$

A **center** has solution formula

$$\begin{aligned}\vec{u}(t) &= \cos(bt)\,\vec{c}_1 + \sin(bt)\,\vec{c}_2,\\ \vec{c}_1 &= \vec{u}(0), \quad \vec{c}_2 = \frac{1}{b}\,A\vec{u}(0).\end{aligned}$$

All solutions are bounded harmonic oscillations of natural frequency b. Orbits in the phase plane are periodic closed curves of period $2\pi/b$ which encircle the origin.

Saddle λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$

A **saddle** has solution formula

$$\begin{aligned}\vec{u}(t) &= e^{\lambda_1 t}\vec{c}_1 + e^{\lambda_2 t}\vec{c}_2,\\ \vec{c}_1 &= \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\,\vec{u}(0), \quad \vec{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\,\vec{u}(0).\end{aligned}$$

The phase portrait shows two lines through the origin which are tangents at $t = \pm\infty$ for all orbits.
The line directions are given by the eigenvectors of matrix A. See Figure 13.

Node λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$

The solution formulas are

$$\begin{aligned}\vec{u}(t) &= e^{\lambda_1 t}\left(\vec{a}_1 + t\vec{a}_2\right), \quad \text{when} \quad \lambda_1 = \lambda_2,\\ \vec{a}_1 &= \vec{u}(0), \quad \vec{a}_2 = (A - \lambda_1 I)\vec{u}(0),\end{aligned}$$

$$\begin{aligned}\vec{u}(t) &= e^{\lambda_1 t}\vec{b}_1 + e^{\lambda_2 t}\vec{b}_2, \quad \text{when} \quad \lambda_1 \neq \lambda_2,\\ \vec{b}_1 &= \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\,\vec{u}(0), \quad \vec{b}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\,\vec{u}(0).\end{aligned}$$

Proper Node (a.k.a. **Star Node)**. Matrix A is required to have two eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$ with $\lambda_1 = \lambda_2$. Then $\vec{u}(0)$ in $\mathbf{span}(\vec{v}_1, \vec{v}_2)$ implies $\vec{u}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{a}_2 = (A - \lambda_1 I)\vec{u}(0) = \vec{0}$. Therefore, $\vec{u}'(t)/|\vec{u}'(t)| = \pm\vec{u}(0)/|\vec{u}(0)|$ implies trajectories are tangent to the line through $(0, 0)$ in direction $\vec{v} = \vec{u}(0)/|\vec{u}(0)|$. Because $\vec{u}(0)$ is arbitrary, $\vec{v}$ can be any direction, which explains the star-like phase portrait in Figure 14

Improper Node with One Eigenpair (a.k.a. **Degenerate Node)**. Matrix A is required to have just one eigenpair $(\lambda_1, \vec{v}_1)$ and $\lambda_1 = \lambda_2$. Then $\vec{u}'(t) = (\vec{a}_2 + \lambda_1\vec{a}_1 + t\lambda_1\vec{a}_2)e^{\lambda_1 t}$ implies $\vec{u}'(t)/|\vec{u}'(t)| \approx \vec{a}_2/|\vec{a}_2|$ at $|t| = \infty$. Matrix $A - \lambda_1 I$ has rank 1, hence $\mathbf{Image}(A - \lambda_1 I) = \mathbf{span}(\vec{v})$ for some nonzero vector $\vec{v}$. Then $\vec{a}_2 = (A - \lambda_1 I)\vec{u}(0)$ is a multiple of $\vec{v}$. Trajectory $\vec{u}(t)$ is tangent to the line through $(0, 0)$ with direction $\vec{v}$, as in Figure 15.

Improper Node with Two Distinct Eigenvalues. Discussed here is the first possibility when matrix A has real eigenvalues with $\lambda_2 < \lambda_1 < 0$. The second possibility $\lambda_2 > \lambda_1 > 0$ is left to the reader. Then $\vec{u}'(t) = \lambda_1\vec{b}_1 e^{\lambda_1 t} + \lambda_2\vec{b}_2 e^{\lambda_2 t}$ implies $\vec{u}'(t)/|\vec{u}'(t)| \approx \vec{b}_1/|\vec{b}_1|$ at $t = \infty$. In terms of eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$, we compute $\vec{b}_1 = c_1\vec{v}_1$ and $\vec{b}_2 = c_2\vec{v}_2$ where $\vec{u}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$. Trajectory $\vec{u}(t)$ is tangent to the line through $(0, 0)$ with direction $\vec{v}_1$. See Figure 15.

Attractor and Repeller

An equilibrium point is called an **attractor** provided orbits starting nearby limit to the point as $t \to \infty$. A **repeller** is an equilibrium point such that orbits starting nearby limit to the point as $t \to -\infty$. Terms like **attracting node** and **repelling spiral** are defined analogously.

Linear Classification Shortcut for $\frac{d}{dt}\vec{u} = A\vec{u}$

Presented here is a practical method for deciding the classification of center, spiral, saddle or node for a linear system $\frac{d}{dt}\vec{u} = A\vec{u}$. The method uses just the eigenvalues of A and the corresponding Euler atoms.

Cayley-Hamilton Basis.

A system $\frac{d}{dt}\vec{u} = A\vec{u}$ will have general solution

$$\vec{u} = \vec{d}_1(\text{Euler Atom 1}) + \vec{d}_2(\text{Euler Atom 2}).$$

The vectors $\vec{d}_1, \vec{d}_2$ depend on A and $\vec{u}(0)$. They are never explicitly used in the shortcut, hence never computed.

The two Euler solution atoms are found from roots λ of the characteristic equation $|A - \lambda I| = 0$. There are two kinds of atoms:

> Sine and cosine appear in the atoms, which make a **rotating** phase portrait, which is either a center or a spiral.
>
> No sine or cosine appear in the atoms, making a **non-rotating** phase portrait, which is either a node or a saddle.

Table 1. Rotating Phase Portraits

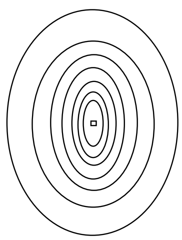

Figure 16. Center

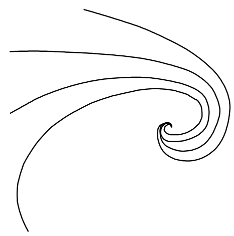

Figure 17. Spiral

Table 2. Non-Rotating Phase Portraits

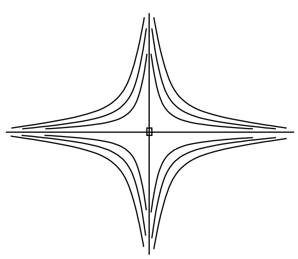

Figure 18. Saddle

Euler solution atoms for a saddle or node have form e^{at}, e^{bt} or else e^{at}, te^{at}. There are no sine or cosine terms.

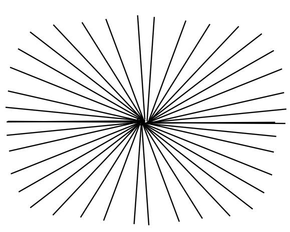

Figure 19. Proper node

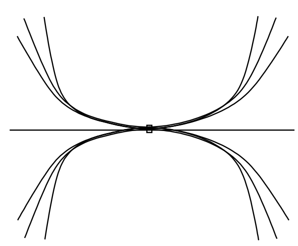

Figure 20. Improper node

Divide and Conquer. Given 2×2 matrix A with $|A| \neq 0$, find the roots of the characteristic equation $|A - \lambda I| = 0$ and construct the two Euler solution atoms. The classification figure, selected from center, spiral, node, saddle, depends only on the atoms. Examine the atoms for sines and cosines. If present, then it will be a rotating figure (center, spiral), otherwise it will be a non-rotating figure (node, saddle). One more divide and conquer decides the figure, because within each figure group, rotating or non-rotating, there is only one attractor/repeller.

> **Rotation Test**. Suppose sines and cosines appear in the Euler atoms. If the Euler atoms are pure sine and cosine, then $(0, 0)$ is a **center**, otherwise $(0, 0)$ is a **spiral**.
>
> **Non-Rotation Test**. Suppose no sines or cosines appear in the Euler atoms. If at $t = \infty$ one Euler atom limits to zero and the other Euler atom limits to infinity, then $(0, 0)$ is a **saddle**, otherwise it is a **node**.

Stability Classification by Euler Atoms.

> A center is always stable, characterized by Euler atoms being pure sine and cosine.
>
> If $(0, 0)$ is not a center, then $(0, 0)$ is stable at $t = \infty$ if and only if both Euler atoms limit to zero at $t = \infty$.

Divide and conquer via Euler atoms requires no table to decide upon the basic phase portrait classification: spiral, center, saddle, node. Stability is likewise decided by Euler atoms.

Node Sub-classifications

If finer geometric sub-classifications of a node are useful to you, then eigenanalysis is required. Assumed below are λ_1, λ_2 real and $\lambda_1\lambda_2 > 0$. *Diagonalizable* means there are two eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$.

Node with Equal Eigenvalues

There are two sub-classifications for a matrix A with real equal eigenvalues $\lambda_1 = \lambda_2$. The directions referenced below are provided by the span of the eigenvectors, which is either 2-dimensional (all directions possible) or 1-dimensional (just two directions possible).

> **Star Node**: Matrix A is diagonalizable with $\lambda_1 = \lambda_2 \neq 0$. Equilibrium $(0, 0)$ is an attractor (or a repeller) from all directions.
>
> **Degenerate Node**: Matrix A is not diagonalizable and $\lambda_1 = \lambda_2 \neq 0$. Equilibrium $(0, 0)$ is an attractor (or a repeller) from directions $\pm\vec{v}_1$, where $(\lambda_1, \vec{v}_1)$ is the only eigenpair.

Node with Unequal Eigenvalues

Matrix A two eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$, because $\lambda_1 \neq \lambda_2$. Equilibrium $(0,0)$ is an attractor (or a repeller) from directions $\pm\vec{v}$, where $\vec{v}$ is one of the two eigenvectors.

Proper Node and Improper Node Classifications

The classifications **proper** and **improper** organize the possible node phase portraits according to attractor (or repeller) directions. This terminology may appear in dynamical system literature.

> **Proper Node**: The equilibrium is an attractor (or repeller) from all directions. The phase portrait is a *star node*.
>
> **Improper Node**: The equilibrium is an attractor (or repeller) from only two directions. The phase portraits include everything except the star node, which includes a *degenerate node* and a *node with unequal eigenvalues*.

How to sort out the terminology? The rule is: **proper** = **star**. Every non-star node is **improper**.

Examples and Methods

1 **Example (Spiral)** Show the classification details for the spirals represented by the matrices

$$\begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}.$$

Solution: Matrix $\begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}$ has characteristic equation $(\lambda - 5)^2 + 4 = 0$. Then $\lambda = 5 \pm 2i$ and the Euler atoms are $e^{5t}\cos(2t), e^{5t}\sin(2t)$. The atoms have sines and cosines, which limits the classification to a center or a spiral. The presence of the exponential factor e^{5t} implies it is not a center, therefore it is a spiral. Because the atoms limit to zero at $t = -\infty$, then $(0,0)$ is a repeller. Classification: unstable spiral.

Matrix $\begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$ has characteristic equation $(\lambda+1)^2+9 = 0$. Then $\lambda = -1\pm 3i$ and the Euler atoms are $e^{-t}\cos(3t), e^{-t}\sin(3t)$. The atoms have sines and cosines, which implies rotation, either a center or a spiral. The presence of the exponential factor e^{-t} implies it is not a center, therefore it is a spiral. Because the atoms limit to zero at $t = \infty$, then $(0,0)$ is an attractor. Classification: stable spiral.

2 **Example (Center)** Matrix $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ represents a center. Show the classification details.

Solution: The characteristic equation $\lambda^2 + 4 = 0$ has complex roots $\lambda = \pm 2i$. The Euler atoms are $\cos(2t), \sin(2t)$, therefore a rotating figure is expected. Because of pure sines and cosines and no exponentials, the initial classification of spiral or center reduces to a center. Always a center is stable. Classification: stable center.

3 Example (Saddle) Show the classification details for the saddles represented by the matrices $\begin{pmatrix} 5 & 4 \\ 10 & 1 \end{pmatrix}$, $\begin{pmatrix} -5 & 4 \\ 2 & 1 \end{pmatrix}$.

Solution: We'll use the theorem $|A - \lambda I| = \lambda^2 + \mathbf{trace}(A)(-\lambda) + |A|$ to find the characteristic equation. Symbol $\mathbf{trace}(A)$ is the sum of the diagonal elements of A and symbol $|A|$ is the determinant of A, evaluated by Sarrus's rule.

The characteristic equations are

$$\lambda^2 - 6\lambda - 35 = 0, \quad \lambda^2 + 4\lambda - 13 = 0.$$

The roots are $3 \pm 2\sqrt{11}$ $(9.6, -3.6)$ and $-2 \pm \sqrt{17}$ $(2.1, -6.1)$, respectively. Therefore, the roots a, b are real with $a > 0$ and $b < 0$. Euler atoms are e^{at}, e^{bt}. The absence of sines and cosines implies the equilibrium $(0,0)$ is non-rotating, either a saddle or a node. Because one atom limits to ∞ and the other to zero, at $t = \pm\infty$, then $(0,0)$ is a saddle. A saddle is always unstable. Classifications: $(0,0)$ is an unstable saddle for both matrices.

4 Example (Node Sub-Classification: Equal Eigenvalues) Show the node classification details for the matrices $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$.

Solution: A 2×2 matrix is called **diagonalizable** provided it has 2 eigenpairs. Then $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ is diagonalizable whereas $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ is not diagonalizable.

The eigenvalues of both matrices are $5, 5$. Euler atoms are the same for both matrices: e^{5t}, te^{5t}. The absence of sines and cosines limits the classification to saddle or node. Because these atoms limit to zero at $t = -\infty$, then $(0,0)$ is a node. For both, $(0,0)$ is a repeller.

The repeller directions are provided by the span of the eigenvectors, which is either 2-dimensional (all directions possible) or 1-dimensional (just two directions possible). See page 700.

The repeller directions for $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ are $\mathbf{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \mathcal{R}^2$ (all directions).

The repeller directions for $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ are $\mathbf{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, which implies just two unit directions $\pm\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Classifications: $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ is an **unstable proper node** (*star node*) and $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ is an **unstable improper node** (*degenerate node*).

5 Example (Node Sub-Classification: Unequal Eigenvalues) Show the node classification details for the matrices $\begin{pmatrix} -5 & 0 \\ 0 & -7 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$.

Solution: Both matrices are diagonal, hence each has two independent eigenvectors. This example shows that diagonalizability by itself does not decide a node sub-classification.

Matrix $\begin{pmatrix} -5 & 0 \\ 0 & -7 \end{pmatrix}$ has unequal eigenvalues $-5, -7$ with Euler atoms e^{-5t}, e^{-7t}. Absence of sines and cosines limits the classification to saddle or node. The atoms have limit zero at $t = \infty$, which eliminates the saddle classification. Therefore, $(0, 0)$ is an attractor. Classification: stable node.

For $\begin{pmatrix} -5 & 0 \\ 0 & -7 \end{pmatrix}$, an attractor orbit is tangent at $t = \infty$ to $\pm\vec{v}_1$, where $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a unit eigenvector for $\lambda_1 = -5$.

Matrix $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$ has unequal eigenvalues $5, 7$ with Euler atoms e^{5t}, e^{7t}. Absence of sines and cosines limits the classification to saddle or node. The atoms have limit zero at $t = -\infty$, which eliminates the saddle classification. Therefore, $(0, 0)$ is a repeller. Classification: unstable node.

For $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$, a repeller orbit is tangent at $t = \infty$ to $\pm\vec{v}_2$, where $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a unit eigenvector for $\lambda_2 = 7$.

Computer Phase Diagrams. In computer node plots for unequal eigenvalues, an eigenvector direction can be detected from limits at $t = \pm\infty$. Attractors will have the eigenvector direction for eigenvalue λ with $|\lambda|$ smallest. Repellers will have the eigenvector direction for eigenvalue λ with $|\lambda|$ largest.

Exercises 10.2

Planar Constant Linear Systems.

1. (Picard's Theorem) Explain why planar solutions don't cross, by appeal to Picard's existence-uniqueness theorem for $\frac{d}{dt}\vec{u} = A\vec{u}$.

2. (Equilibria) System $\frac{d}{dt}\vec{u} = A\vec{u}$ always has solution $\vec{u}(t) = \vec{0}$, so there is always one equilibrium point. Give an example of a matrix A for which there are infinitely many equilibria.

Putzer's Formula.

3. (Cayley-Hamilton) Define matrices $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, expand left and right sides to verify the **Cayley-Hamilton identity**
$A^2 - (c + d)A + (ad - bc)\mathbf{I} = \mathbf{0}$.

4. (Complex Roots) Verify the Putzer solution $\vec{u} = \Phi(t)\vec{u}(0)$ of $\vec{u}' = A\vec{u}$ for complex roots $\lambda_1 = \overline{\lambda_2} = a + bi$, $b > 0$, where $\Phi(t)$ is

$$e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right).$$

5. (Distinct Eigenvalues) Solve

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\vec{u}.$$

6. (Real Equal Eigenvalues) Solve

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}\vec{u}.$$

7. (Complex Eigenvalues) Solve

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \vec{u}.$$

Continuity and Redundancy.

8. (Real Equal Eigenvalues) Show that limiting $\lambda_2 \to \lambda_1$ in the Putzer formula for distinct eigenvalues gives Putzer's formula for real equal eigenvalues.

9. (Complex Eigenvalues) Assume $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$. Then Putzer's first formula holds. Show the third formula details for $\Phi(t)$:

$$e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right).$$

Illustrations.

10. (Distinct Eigenvalues) Show the details for the solution of

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix} \vec{u}.$$

11. (Complex Eigenvalues) Show the details for the solution of

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix} \vec{u}.$$

Isolated Equilibria.

12. (Determinant Expansion) Verify that $|A - \lambda I|$ equals

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

13. (Infinitely Many Equilibria) Explain why $A\vec{u} = \vec{0}$ has infinitely many solutions when $\det(A) = 0$.

Classification of Equilibria.

14. (Rotating Figures) When sines and cosines appear in the Euler atoms, the phase portrait at $(0, 0)$ rotates around the origin. Explain precisely why this is true.

15. (Non-Rotating Figures) When sines and cosines do not appear in the Euler atoms, the phase portrait at $(0, 0)$ has no rotation. Give a precise explanation.

Attractor and Repeller.

16. (Classification) Which of spiral, center, saddle, node can be an attractor or a repeller?

17. (Attractor) Prove that $(0, 0)$ is an attractor if and only if the Euler atoms have limit zero at $t = \infty$.

18. (Repeller) Prove that $(0, 0)$ is a repeller if and only if the Euler atoms have limit zero at $t = -\infty$.

19. (Center) A center is neither an attractor nor a repeller. Explain, using Euler atoms.

Phase Portrait Linear.

Show the classification details for spiral, center, saddle, proper node, improper node. Include a drawing which identifies eigenvector directions, where such information applies.

20. (Spiral)

$$\begin{aligned} \tfrac{d}{dt}x &= 2x + 3y, \\ \tfrac{d}{dt}y &= -3x + 2y. \end{aligned}$$

21. (Center)

$$\begin{aligned} \tfrac{d}{dt}x &= 3y, \\ \tfrac{d}{dt}y &= -3x. \end{aligned}$$

22. (Saddle)

$$\begin{aligned} \tfrac{d}{dt}x &= 3x, \\ \tfrac{d}{dt}y &= -5y. \end{aligned}$$

23. (Proper Node)

$$\begin{aligned} \tfrac{d}{dt}x &= 2x, \\ \tfrac{d}{dt}y &= 2y. \end{aligned}$$

24. (Improper Node: Degenerate)

$$\begin{aligned} \tfrac{d}{dt}x &= 2x + y, \\ \tfrac{d}{dt}y &= 2y. \end{aligned}$$

25. (Improper Node: $\lambda_1 \neq \lambda_2$)

$$\begin{aligned} \tfrac{d}{dt}x &= 2x + y, \\ \tfrac{d}{dt}y &= 3y. \end{aligned}$$

10.3 Planar Almost Linear Systems

A nonlinear planar autonomous system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is called **almost linear** at equilibrium point $\vec{u} = \vec{u}_0$ if

$$\vec{F}(\vec{u}) = A(\vec{u} - \vec{u}_0) + \vec{G}(\vec{u}),$$
$$\lim_{\|\vec{u}-\vec{u}_0\|\to 0} \frac{\|\vec{G}(\vec{u})\|}{\|\vec{u} - \vec{u}_0\|} = 0.$$

The function $\vec{G}$ has the same smoothness as $\vec{F}$. We investigate the possibility that a local phase portrait at $\vec{u} = \vec{u}_0$ for the nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is graphically identical to the one for the linear system $\vec{v}'(t) = A\vec{v}(t)$ at $\vec{v} = 0$.

The results will apply to **all isolated equilibria** of $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$. This is accomplished by expanding F in a Taylor series about each equilibrium point, which implies that the ideas are applicable to different choices of A and G, depending upon which equilibrium point $\vec{u}_0$ was considered.

Define the **Jacobian matrix** of $\vec{F} = \begin{pmatrix} f \\ g \end{pmatrix}$ at equilibrium point $\vec{u}_0$ by the formula

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}.$$

Taylor's theorem for functions of two variables says that

$$\vec{F}(\vec{u}) = J(\vec{u} - \vec{u}_0) + \vec{G}(\vec{u})$$

where $\vec{G}(\vec{u})/\|\vec{u}-\vec{u}_0\| \to 0$ as $\|\vec{u}-\vec{u}_0\| \to 0$. Therefore, for $\vec{F}$ continuously differentiable, we may always take $A = J$ to obtain from the almost linear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ its **linearization** $\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$.

Phase Portrait of an Almost Linear System

For planar almost linear systems $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$, phase portraits have been studied extensively, by Poincaré-Bendixson and a long list of researchers. It is known that only a finite number of local phase portraits are possible near each isolated equilibrium point of the nonlinear system, the library of figures being identical to those possibilities for a linear system $\vec{v}'(t) = A\vec{v}(t)$. A precise statement, without proof, appears below.

Theorem 4 (Paste Theorem: Almost Linear System Phase Portrait)
Let the planar almost linear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ be given with $\vec{F}(\vec{u}) = A(\vec{u} - \vec{u}_0) + \vec{G}(\vec{u})$ near the isolated equilibrium point $\vec{u}_0$ (an isolated root of $\vec{F}(\vec{u}_0) = \vec{0}$ with $|A| \neq 0$). Let λ_1, λ_2 be the roots of $\det(A - \lambda I) = 0$. Then:

1. If $\lambda_1 = \lambda_2$, then the equilibrium $\vec{u}_0$ of the nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is either a node or a spiral. The equilibrium $\vec{u}_0$ is an asymptotically stable attractor if $\lambda_1 < 0$ and it is a repeller if $\lambda_1 > 0$. In short, the nonlinear system inherits stability from the linear system.

2. If $\lambda_1 = \overline{\lambda}_2 = ib$ with $b > 0$, then the equilibrium $\vec{u}_0$ of the nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is either a center or a spiral. The stability of the equilibrium $\vec{u}_0$ cannot be predicted from properties of A.

3. In all other cases, the isolated equilibrium $\vec{u}_0$ has graphically the same local phase portrait as the associated linear system $\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$ at $\vec{v} = \vec{0}$. In particular, local phase portraits of a saddle, spiral or node can be graphed from the linear system. The nonlinear system inherits locally the linearized system properties of stability and instability.

Classification of Almost Linear Equilibria

A system $\frac{d}{dt}\vec{u}(t) = A\left(\vec{u}(t) - \vec{u}_0\right) + \vec{G}(\vec{u}(t))$ has a local phase portrait determined by the linear system $\vec{v}'(t) = A\vec{v}(t)$, except in the case when the roots λ_1, λ_2 of the characteristic equation $\det(A - \lambda I) = 0$ are equal or purely imaginary (see Theorem 4). To summarize:

Table 3. Equilibria classification for almost linear systems

Eigenvalues of A	**Nonlinear Classification**
$\lambda_1 < 0 < \lambda_2$	Unstable saddle
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral
$\lambda_1 = \overline{\lambda}_2 = a + ib$, $a < 0$, $b > 0$	Stable spiral
$\lambda_1 = \overline{\lambda}_2 = a + ib$, $a > 0$, $b > 0$	Unstable spiral
$\lambda_1 = \overline{\lambda}_2 = ib$, $b > 0$	Stable or unstable, center or spiral

Almost Linear Equilibria Geometry

Applied literature may refer to an equilibrium point $\vec{u}_0$ of a nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ as a spiral, center, saddle or node. The geometry of these classifications is explained below.

Spiral. To describe a **nonlinear spiral**, we require that an orbit starting on a given ray emanating from the equilibrium point must intersect that ray in infinitely many distinct points on $(-\infty, \infty)$.

Intutition. Basic understanding of a **nonlinear spiral** is obtained from a linear example, e.g.,

$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \vec{u}(t).$$

An orbit has component solutions

$$x(t) = e^{-t}(A\cos 2t + B\sin 2t), \quad y(t) = e^{-t}(-A\sin 2t + B\cos 2t)$$

which oscillate infinity often on $(-\infty, \infty)$, rotating around equilibrium point $(0, 0)$ with amplitude Ce^{-t}, for some constant $C > 0$.

Center. Local orbits are periodic solutions. Each local orbit is a closed curve which forms a planar region with boundary, having the equilibrium point interior. As the periodic orbits shrink, the planar region also shrinks, limiting as a planar set to the equilibrium point. Drawings often portray the periodic orbit as a convex figure, but this is not correct, in general, because the periodic orbit can have any shape. In particular, the linearized system may have phase portrait consisting of concentric circles, but the nonlinear phase portrait has no such exact geometric structure.

Saddle. The term implies that *locally* the phase portrait looks like a linear saddle. In nonlinear phase portraits, the straight lines to which orbits are asymptotic appear to be curves instead. These curves are called **separatrices**, which are generally unions of certain orbits and equilibria.

Node. Each orbit starting near the equilibrium is expected to limit to the equilibrium at either $t = \infty$ (stable attractor) or $t = -\infty$ (unstable repeller), in a fashion asymptotic to a direction $\vec{v}$. The terminology applies when the linearized system is a **proper node** (a.k.a. *star node*), in which case there is an orbit asymptotic to $\vec{v}$ for every direction $\vec{v}$. If there is only one direction $\vec{v}$ possible, or all orbits are asymptotic to just one separatrix, then the equilibrium is classified as an **improper node**. The term *degenerate node* applies to a subclass of improper nodes – see Example 4, page 702.

Pasting Figures to make a Nonlinear Phase Portrait

The plan provided by the theorem is to paste a library source figure, one of spiral, center, saddle or node, overlaying $(0, 0)$ in the source figure atop equilibrium point $\vec{u} = \vec{u}_0$ in the nonlinear phase portrait. Some observations follow, about what works and what fails.

1. The local paste is valid to graphical resolution near $\vec{u} = \vec{u}_0$, and invalid far away from the equilibrium point.

2. The pasted figure can mutate into a spiral, if the source figure is either a center, or else a node with $\lambda_1 = \lambda_2$. Otherwise, saddle, spiral and node locally paste into saddle, spiral, node.

3. Stability of the source figure is inherited by the nonlinear portrait, except when the source is a center. In this one exceptional case, no stability conclusion can be drawn. However, an attractor or repeller source figure always pastes into an attractor or a repeller.

Examples and Methods

6 Example (Compute Isolated Equilibria) Find all equilibria for the nonlinear system

$$x'(t) = x(t) + y(t), \quad y'(t) = 1 - x^2(t).$$

Solution: Equilibria are constant solutions, obtained formally by setting $x' = y' = 0$ in the two differential equations $x' = x + y, y' = 1 - x^2$. Then solve for constants x, y. The details:

Set $x' = 0$	$0 = x + y$
Set $y' = 0$	$0 = 1 - x^2$
Solve for x, y	$x = \pm 1$, $y = -x$.
Equilibria	$(1, -1)$ and $(-1, 1)$

7 Example (Linearization at Equilibria) Find the two linearizations at equilibria $(1, -1), (-1, 1)$ for the nonlinear system

$$x'(t) = x(t) + y(t), \quad y'(t) = 1 - x^2(t).$$

Solution: The system of differential equations is written with function notation in the form $x' = f(x, y), y' = g(x, y)$. Then

$$f(x, y) = x + y, \quad g(x, y) = 1 - x^2.$$

The Jacobian matrix

$$J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

is computed with symbols x, y, f, g as follows.

Partial derivative $f_x(x, y)$:	$f_x = \partial_x(x + y) = 1 + 0 = 1$
Partial derivative $g_x(x, y)$:	$g_x = \partial_x(1 - x^2) = 0 - 2x = -2x$
Partial derivative $f_y(x, y)$:	$f_y = \partial_y(x + y) = 0 + 1 = 1$
Partial derivative $g_y(x, y)$:	$g_y = \partial_y(1 - x^2) = 0 - 0 = 0$

Then

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2x & 0 \end{pmatrix}.$$

The symbols x, y are used for the two substitutions: $x = 1, y = -1$ and $x = -1, y = 1$.

$$J(1,-1) = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}, \quad J(-1,1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

The two linearized problems are

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\vec{u}, \quad \frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\vec{u}.$$

8 Example (Classification of Linearized Problems) Classify the two linear problems

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\vec{u}, \quad \frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\vec{u}.$$

Solution:

The answers: $\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$ is an unstable spiral; $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ is an unstable saddle.

The two characteristic equations are $\lambda^2 - \lambda + 2 = 0$ and $\lambda^2 + \lambda + 2 = 0$ with roots, respectively, $\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$ and $2, -1$. According to the classification theory, page 696, the equilibrium $(0,0)$ is respectively an unstable spiral or an unstable saddle.

9 Example (Pasting Library Linear Portraits onto Nonlinear Portraits) Classify equilibria $(1,-1), (-1,1)$ for the nonlinear system

$$x'(t) = x(t) + y(t), \quad y'(t) = 1 - x^2(t),$$

as *nonlinear* spiral, center, saddle or node. Paste the linear portraits for $J(-1,1), J(1,-1)$ onto the nonlinear direction field portrait, when possible.

Solution: **Classifications**: $(-1,1)$ is a nonlinear unstable saddle; $(1,-1)$ is a nonlinear unstable spiral.

Previous examples show that for the linearized problems, $(-1,1)$ is an unstable saddle and $(-1,1)$ is an unstable spiral. Theorem 4 applies to conclude that the two linear phase portraits directly transfer onto the nonlinear phase portrait. This means that $(0,0)$ in each source figure can be pasted atop the corresponding equilibrium point in the nonlinear system, the pasted figure valid locally.

Computer phase portraits show the two pasted library figures with automatic fine tuning. Especially, the saddle will be tuned, because a library source figure usually has asymptotes parallel to the coordinate axes, whereas the computer graphic will show tuned asymptotes in eigenvector directions.

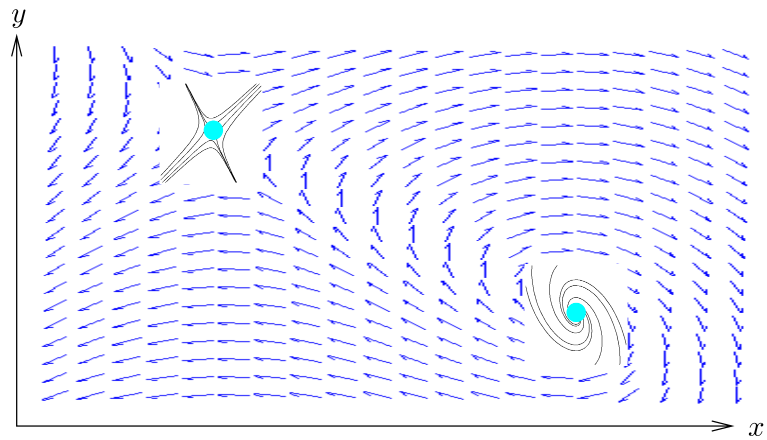

Figure 21. Pasting Source Figures onto a Nonlinear Phase portrait. Saddle at $(-1,1)$, spiral at $(1,-1)$. The saddle source uses a linear phase portrait for $\frac{d}{dt}\vec{v} = J(-1,1)\vec{v}$. The standard saddle source can be rotated to match the nonlinear direction field, with a similar result.

10 Example (Trout System) Consider a trout model for two species x, y:

$$\begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

The equilibria are $(0,0), (90,0), (0,60), (80,20)$. Find the linearized problem for each equilibrium, then make a tuned computer plot.

Solution:

System Form. Let $f(x,y) = x(-2x-y+180), g(x,y) = y(-x-2y+120)$ to convert to system form $x' = f(x,y), y' = g(x,y)$.

Jacobian Matrix. Use symbols f, g, x, y to compute the Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$.

$f_x = \frac{\partial}{\partial x}\left(-2x^2 - xy + 180x\right) = -4x - y - 180$

$f_y = \frac{\partial}{\partial y}\left(-2x^2 - xy + 180x\right) = -x$

$g_x = \frac{\partial}{\partial x}\left(-xy - 2y^2 + 120y\right) = -y$

$g_y = \frac{\partial}{\partial y}\left(-xy - 2y^2 + 120y\right) = -x - 4y + 120$

$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -4x - y - 180 & -x \\ -y & -x - 4y + 120 \end{pmatrix}$

Equilibria. To find the equilibria, formally set $x' = y' = 0$. Details:

$x' = 0 = f(x,y)$ becomes $x(-2x - y + 180) = 0$

$y' = 0 = g(x,y)$ becomes $y(-x - 2y + 120) = 0$

Set the factors to zero, in four possible ways, to obtain the solutions

$$x = y = 0, \quad x = 0, y = 60, \quad x = 90, y = 0, \quad x = 80, y = 20.$$

Linearized Differential Equations. The linear problems $\frac{d}{dt}\vec{u} = J(x_0, y_0)\vec{u}$ at equilibria $(0, 0), (0, 60), (90, 0), (80, 20)$ are created from the four Jacobian matrices

$$J(0,0) = \begin{pmatrix} -180 & 0 \\ 0 & 120 \end{pmatrix}, \qquad J(0,60) = \begin{pmatrix} 120 & 0 \\ -60 & -120 \end{pmatrix},$$
$$J(90,0) = \begin{pmatrix} -180 & -90 \\ 0 & 30 \end{pmatrix}, \qquad J(80,20) = \begin{pmatrix} -160 & -80 \\ -20 & -40 \end{pmatrix}.$$

Eigenvalues. Answers for the four matrices are respectively $120, 180, \quad 120, -120, 30, -180, \quad -27.89, -172.11$.

Linear Classifications. Because there are no complex eigenvalues, then the possible linear phase portraits are either saddle or node. Checking limits of Euler atoms at $t = \infty$ reveals the classifications unstable node, saddle, saddle, stable node. No equal eigenvalues implies both nodes are **improper**.

Paste Theorem. All linear source figures paste directly onto the nonlinear phase portrait with stability properties inherited. See Theorem 4.

Eigenvectors help understanding of the phase portrait. In all four figures, asymptote directions are along an eigenvector. For instance, at $(80, 20)$ the two eigenvector directions are $\vec{v}_1 = \begin{pmatrix} -0.6 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6.6 \\ 1 \end{pmatrix}$.

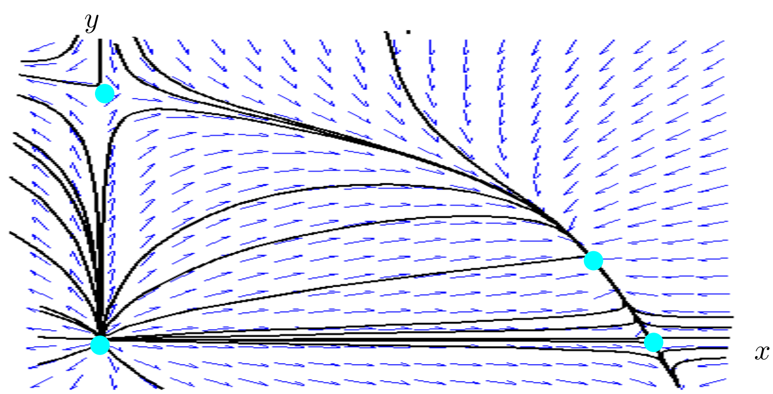

Figure 22. Trout System Phase portrait.
Saddles at $(0, 60)$ and $(90, 0)$. Improper nodes with unequal eigenvalues at $(0, 0)$ and $(80, 20)$. A separatrix can be visualized, which connects $(90, 0)$ to $(0, 0)$ to $(60, 0)$ along the coordinate axes, and then to $(80, 20)$.

11 Example (Rabbit-Fox System) Consider a predator-prey model for rabbits $x(t)$ and foxes $y(t)$:

$$\begin{aligned} x' &= \frac{1}{200}x(40 - y), \\ y' &= \frac{1}{100}y(x - 50). \end{aligned}$$

The equilibria are $(0, 0), (50, 40)$. Find the linearized problem for each equilibrium, then make a tuned computer plot.

Solution:
System Form. Let $f(x,y) = \frac{1}{200}x(40-y), g(x,y) = \frac{1}{100}y(x-50)$ to convert to system form $x' = f(x,y), y' = g(x,y)$.

Jacobian Matrix. Symbols f, g, x, y are used to compute the Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$.

$f_x = \frac{\partial}{\partial x}(x/5 - xy/200) = 1/5 - y/200$

$f_y = \frac{\partial}{\partial y}(x/5 - xy/200) = -x/200$

$g_x = \frac{\partial}{\partial x}(-y/2 + xy/100) = y/100$

$g_y = \frac{\partial}{\partial y}(-y/2 + xy/100) = -x - 4y + 120$

$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -4x - y - 180 & -x \\ -y & -x - 4y + 120 \end{pmatrix}$

Equilibria. To find the equilibria $(0,0), (50,40)$, formally set $x' = y' = 0$. Details:

$0 = f(x,y)$ becomes $\frac{1}{200}x(40-y) = 0$

$0 = g(x,y)$ becomes $\frac{1}{100}y(x-50) = 0$

The solutions are $x = y = 0$ or else $x = 50, y = 40$.

Linearized Differential Equations. The linear problems $\frac{d}{dt}\vec{u} = J(x_0, y_0)\vec{u}$ at equilibria $(0,0), (50,40)$ are created from the two Jacobian matrices

$$J(0,0) = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad J(50,40) = \begin{pmatrix} 0 & -\frac{1}{4} \\ \frac{2}{5} & 0 \end{pmatrix}.$$

Eigenvalues. The answers are $\frac{1}{5}, -\frac{1}{2}$ and $\pm i/\sqrt{10}$, respectively.

Linear Classifications. Complex eigenvalues imply linear phase portraits of either center or node. Checking Euler atoms reveals the classification **center** at $(50,40)$. Real unequal eigenvalues at $(0,0)$ implies a saddle or node. Checking limits of the Euler atoms at $t = \infty$ implies $(0,0)$ is a **saddle**. Both linear source figures are **stable**.

Paste Theorem. The linear saddle source figure for $(0,0)$ pastes directly onto the nonlinear phase portrait at $(0,0)$ with stability properties inherited. The linear center source figure for $(50,40)$ pastes into a center or a spiral at $(50,40)$. The paste stability or instability is not decided. See Theorem 4.

The easiest path to deciding the nonlinear portrait at $(50,40)$ is a computer phase portrait, which shows a center structure.

Eigenvectors help understanding of the phase portrait. At $(0,0)$ the two eigenvector directions are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

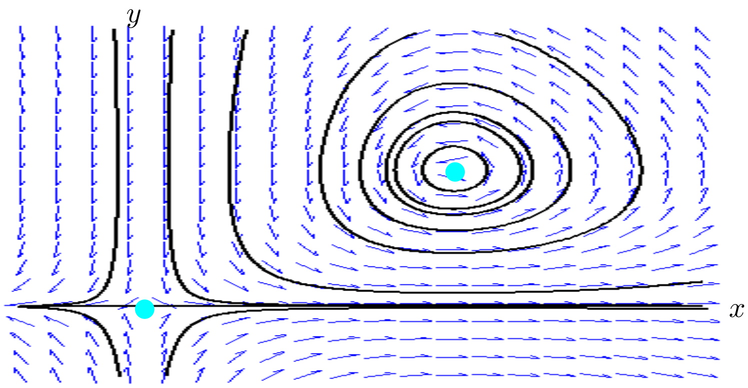

Figure 23. Rabbit-Fox System Phase portrait.
Eigenvector directions for the saddle at $(0,0)$ are parallel to the coordinate axes. The linear center from $J(50,40)$ happens to transfer to a nonlinear center at $(50,40)$.

Exercises 10.3

Almost Linear Systems. Find all equilibria (x_0, y_0) of the given nonlinear system. Then compute the Jacobian matrix $A = J(x_0, y_0)$ for each equilibria.

1. (Spiral and Saddle)

$$\begin{aligned} \frac{d}{dt}x &= x + 2y, \\ \frac{d}{dt}y &= 1 - x^2. \end{aligned}$$

2. (Saddle and Two Spirals)

$$\begin{aligned} \frac{d}{dt}x &= x - 3y + 2xy, \\ \frac{d}{dt}y &= 4x - 6y - xy - x^2. \end{aligned}$$

3. (Spiral, Saddle)

$$\begin{aligned} \frac{d}{dt}x &= 3x - 2y - x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y. \end{aligned}$$

4. (Center and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

5. (Proper Node and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= y - xy. \end{aligned}$$

6. (Improper Degenerate Node, Spiral and Two Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3 + y^3, \\ \frac{d}{dt}y &= y + 3xy. \end{aligned}$$

7. (Improper Node and a Saddle)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

8. (Proper Node and a Saddle)

$$\begin{aligned} \frac{d}{dt}x &= 2x + y^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Phase Portrait Almost Linear. Linear library phase portraits can be locally pasted atop the equilibria of an almost linear system, with limitations. Apply the theory for the following examples. Complete the phase diagram by computer, thereby resolving the possible mutation of a center or node into a spiral. Label eigenvector directions, where it makes sense.

9. (Center and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

10. (Proper Node and 3 Saddles)

$$\begin{aligned}\tfrac{d}{dt}x &= x - y + x^2 - y^2, \\ \tfrac{d}{dt}y &= y - xy.\end{aligned}$$

11. (Improper Degenerate Node, Spiral and Two Saddles)

$$\begin{aligned}\tfrac{d}{dt}x &= x - y + x^3 + y^3, \\ \tfrac{d}{dt}y &= y + 3xy.\end{aligned}$$

12. (Improper Node and a Saddle)

$$\begin{aligned}\tfrac{d}{dt}x &= x - y + x^3, \\ \tfrac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

13. (Proper Node and a Saddle)

$$\begin{aligned}\tfrac{d}{dt}x &= 2x + y^3, \\ \tfrac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

Classification of Almost Linear Equilibria. With computer assist, find and classify the nonlinear equilibria.

14. (Co-existing Species)

$$\begin{aligned}x'(t) &= x(t)(24 - 2x(t) - y(t)), \\ y'(t) &= y(t)(30 - 2y(t) - x(t)).\end{aligned}$$

15. (Doomsday-Extinction)

$$\begin{aligned}x'(t) &= x(t)(x(t) - y(t) - 4), \\ y'(t) &= y(t)(x(t) + y(t) - 8).\end{aligned}$$

Almost Linear Geometry. A separatrix is a union of curves and equilibria with orbits limiting to it. With computer assist, make a plot of threaded curves which identify one or more separatrices near the equilibrium.

16. (Saddle $(-1, 1)$)

$$\begin{aligned}\tfrac{d}{dt}x &= x + y, \\ \tfrac{d}{dt}y &= 1 - x^2.\end{aligned}$$

17. (Saddle $(-1/5, -2/5)$)

$$\begin{aligned}\tfrac{d}{dt}x &= 3x - 2y - x^2 - y^2, \\ \tfrac{d}{dt}y &= 2x - y.\end{aligned}$$

18. (Saddle $(-2/3, \sqrt[3]{4/3})$)

$$\begin{aligned}\tfrac{d}{dt}x &= 2x + y^3, \\ \tfrac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

19. (Degenerate Improper Node)

$$\begin{aligned}\tfrac{d}{dt}x &= x - y + x^3 + y^3, \\ \tfrac{d}{dt}y &= y + 3xy, \quad \text{at } (0, 0).\end{aligned}$$

Rayleigh and van der Pol. Each example below has a unique periodic orbit surrounding an equilibrium point that is the limit at $t = \infty$ of any other orbit. Verify the spiral repeller at $(0, 0)$ in the attached figure, from the linearized problem at $(0, 0)$ and **Paste Theorem** 4. Create phase portraits with computer assist for the linear and nonlinear problems.

20. (Lord Rayleigh 1877, Clarinet Reed Model)

$$\begin{aligned}\tfrac{d}{dt}x &= y, \\ \tfrac{d}{dt}y &= -x + y - y^3.\end{aligned}$$

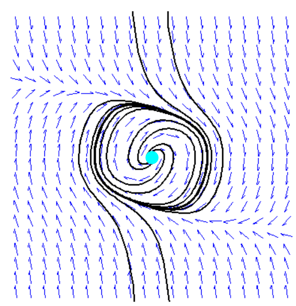

Figure 24. Clarinet Reed.

21. (van der Pol 1924, Radio Oscillator Circuit Model)

$$\begin{aligned}\tfrac{d}{dt}x &= y, \\ \tfrac{d}{dt}y &= -x + (1 - x^2)y.\end{aligned}$$

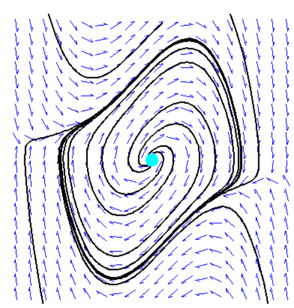

Figure 25. Oscillator Circuit.

10.4 Biological Models

Studied here are **predator-prey models** and **competition models** for two populations. Assumed as background from population biology are the one-dimensional Malthusian model $\frac{d}{dt}P = kP$ and the one-dimensional Verhulst model $\frac{d}{dt}P = (a - bP)P$.

Predator-Prey Models

One species called the **predator** feeds on the other species called the **prey**. The prey feeds on some constantly available food supply, e.g., rabbits eat plants and foxes eat rabbits.

Credited with the classical predator-prey model is the Italian mathematician **Vito Volterra** (1860-1940), who worked on cyclic variations in shark and prey-fish populations in the Adriatic sea. The following biological assumptions apply to model a predator-prey system.

Malthusian Growth	The prey population grows according to the growth equation $x'(t) = a\,x(t)$, $a > 0$, in the absence of predators.
Malthusian Decay	The predator population decays according to the decay equation $y'(t) = -b\,y(t)$, $b > 0$, in the absence of prey.
Chance Encounters	The prey decrease population at a rate $-pxy$, $p > 0$, due to chance encounters of predators y with prey x. Predators increase population due to these chance interactions at a rate qxy, $q > 0$.

The interaction terms qxy and $-pxy$ are justified by arguing that the frequency of chance encounters is proportional to the product xy. Biologists explain the proportionality by saying that doubling either population should double the frequency of chance encounters. Adding the Malthusian rates and the chance encounter rates gives the **Volterra predator-prey system**[1]

$$(1)\qquad \begin{aligned} x'(t) &= (a - p\,y(t))x(t), \\ y'(t) &= (q\,x(t) - b)y(t). \end{aligned}$$

The differential equations are displayed in this form in order to emphasize that each of $x(t)$ and $y(t)$ satisfy a scalar first order differential equation $u'(t) = r(t)u(t)$ in which the rate function $r(t)$ depends on time.

[1]The system is written with prey x and predator y. Alphabetical order **predator-prey** would have y first and then x.

For initial population sizes near zero, the two differential equations behave very much like the Malthusian growth model $u'(t) = a\,u(t)$ and the Malthusian decay model $u'(t) = -b\,u(t)$. This basic growth/decay property allows us to identify the predator variable y, or the prey variable x, regardless of the order in which the differential equations are written. As viewed from Malthus' law $u' = ru$, the prey population has growth rate $r = a - py$ which gets smaller as the number y of predators grows, resulting in fewer prey. Likewise, the predator population has decay rate $r = -b + qx$, which gets larger as the number x of prey grows, causing increased predation. These are the basic ideas of Verhulst, applied to the individual populations x and y.

System Variables

The system of two differential equations (1) can be written as a planar vector autonomous system

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u})$$

where $\vec{F}$ is defined by

$$\vec{F}(\vec{u}) = \begin{pmatrix} (a-py)x \\ (qx-b)y) \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

The vector function $\vec{F}$ is everywhere defined and continuously differentiable. The Picard–Lindelöf theorem provides existence-uniqueness.

A planar vector autonomous system $\frac{d}{dt}\vec{u} = \vec{F}(\vec{u})$ can be written in standard scalar system form

$$x' = f(x,y), \quad y' = g(x,y)$$

by providing definitions for $f(x,y)$ and $g(x,y)$. For predator-prey system (1), the definitions are

$$f(x,y) = (a - p\,y)x, \quad g(x,y) = (q\,x - b)y.$$

Equilibria

The equilibrium points $\vec{u} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ satisfy $\vec{F}(\vec{u}) = \vec{0}$. For predator-prey system (1), the equilibria are $(0,0)$ and $(b/q, a/p)$, found by solving for x_0, y_0 in the equations $(a - p\,y_0)x_0 = 0$, $(q\,x_0 - b)y_0 = 0$.

Linearized Predator-Prey System

The linearized system at equilibrium (x_0, y_0) is the vector-matrix system $\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$, where A is the Jacobian matrix $J(x, y)$ evaluated at point $x = x_0, y = y_0$, briefly $A = J(x_0, y_0)$. In terms of system variables[2],

$$J(x_0, y_0) = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}.$$

For the predator-prey system, we start by computing

$$f_x = \frac{\partial}{\partial x}(a\,x - p\,xy) = a - p\,y, \quad f_y = \frac{\partial}{\partial y}(a\,x - p\,xy) = 0 - p\,x,$$
$$g_x = \frac{\partial}{\partial x}(q\,xy - b\,y) = q\,y - 0, \quad g_y = \frac{\partial}{\partial y}(q\,xy - b\,y) = q\,x - b.$$

The Jacobian matrix is given explicitly by

$$(3) \qquad J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a - p\,y & -p\,x \\ q\,y & q\,x - b \end{pmatrix}.$$

The matrix J is evaluated at equilibrium points $(0, 0), (b/q, a/p)$ to obtain the 2×2 matrices for the linearized systems:

$$J(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}, \quad J(b/q, a/p) = \begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix}.$$

The linearized systems $\vec{v}'(t) = A\vec{v}(t)$ are:

Equilibrium $(0, 0)$ $\qquad \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}\vec{u}(t)$

Equilibrium $(b/q, a/p)$ $\qquad \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix}\vec{u}(t)$

Saddle $J(0, 0)$. Matrix $\begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$ has unequal real eigenvalues $a, -b$ and associated Euler atoms e^{at}, e^{-bt}. No rotation implies a saddle or node, but limits at infinity imply a linear **saddle**. The **Paste Theorem** implies system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ has a saddle at equilibrium $(0, 0)$.

Center $J(b/q, a/p)$. Matrix $\begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix}$ has complex conjugate eigenvalues $\pm i\sqrt{ab}$ and associated Euler atoms $\cos(t\sqrt{ab}), \cos(t\sqrt{ab})$. Pure rotation (no exponential factor) implies a linear **center**. The **Paste Theorem** implies system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ has either a center or a spiral at equilibrium $(b/q, a/p)$.

Shown below in Theorem 5 is that **the spiral case does not happen**. The proof of Lemma 1 is in the exercises.

[2]Notation f_x means $\partial f/\partial x$, the calculus x-derivative with all other variables held constant.

Lemma 1 (Predator-Prey Implicit Solution)
Let $(x(t), y(t))$ be an orbit of the predator-prey system (1) with $x(0) > 0$ and $y(0) > 0$. Then for some constant C,

$$a \ln|y(t)| + b \ln|x(t)| - q\,x(t) - p\,y(t) = C. \tag{4}$$

Theorem 5 (Spiral Case Eliminated)

Equilibrium $(b/q, a/p)$ of predator-prey system (1) cannot be a spiral.

Proof: Assume the equilibrium $(b/q, a/p)$ is a spiral point and some orbit touches the line $x = b/q$ in points $(b/q, u_1)$, $(b/q, u_2)$ with $u_1 \neq u_2$, $u_1 > a/p$, $u_2 > a/p$. Consider the energy function $E(u) = a \ln|u| - p\,u$. Due to relation (4), $E(u_1) = E(u_2) = E_0$, where $E_0 \equiv C + b - b \ln|b/q|$. By the Mean Value Theorem of calculus, $dE/du = 0$ at some u between u_1 and u_2. This is a contradiction, because $dE/du = (a-pu)/u$ is strictly negative for $a/p < u < \infty$. Therefore, equilibrium $(b/q, a/p)$ is **not a spiral**.

Rabbits and Foxes

An instance of predator-prey theory is a Volterra population model for x rabbits and y foxes given by the system of differential equations

$$\begin{aligned} x'(t) &= \frac{1}{250}\,x(t)(40 - y(t)), \\ y'(t) &= \frac{1}{50}\,y(t)(x(t) - 60). \end{aligned} \tag{5}$$

The equilibria of system (5) are $(0, 0)$ and $(60, 40)$. A phase portrait for system (5) appears in Figure 26.

The linearized system at $(60, 40)$ is

$$\begin{aligned} x'(t) &= -\frac{6}{25}y(t), \\ y'(t) &= \frac{4}{5}x(t). \end{aligned}$$

This system has eigenvalues $\pm i\sqrt{24/125}$ and Euler atoms $\sin(t\sqrt{24/125})$, $\cos(t\sqrt{24/125})$, which have period $2\pi/\sqrt{24/125} \approx 14.33934302$. The linear classification is a center.

The nonlinear classification at $(60, 40)$ is then a **center**, because of Theorem 5. Intuition dictates that the period of smaller and smaller nonlinear orbits enclosing the equilibrium $(60, 40)$ must approach a value that is approximately 14.3.

The fluctuations in population size $x(t)$ are measured graphically by the maximum and minimum values of x in the phase portrait, or more simply, by graphing t versus $x(t)$ in a planar graphic. To illustrate, the orbit for $x(0) = 60$, $y(0) = 100$ is graphed in Figure 27, from which it is determined that the rabbit population $x(t)$ fluctuates between 39 and 87. Similar remarks apply to foxes $y(t)$.

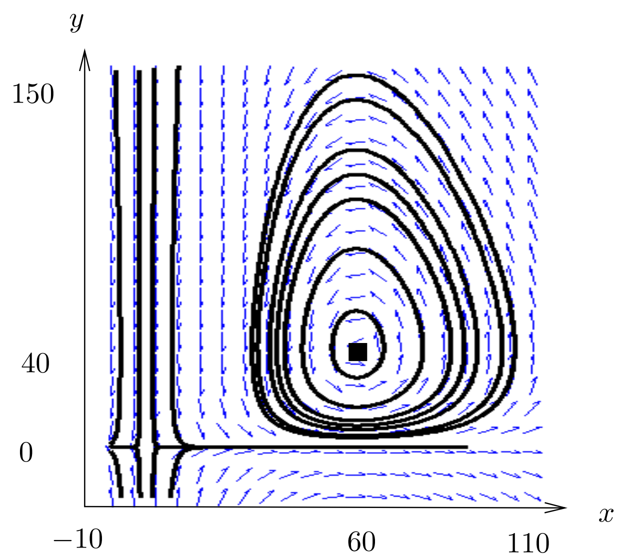

Figure 26. Rabbit and Fox System (5).

Equilibria $(0, 0)$ and $(60, 40)$ are respectively a saddle and a center. The oscillation period is about 17 for the largest orbit and 14.5 for the smallest orbit.

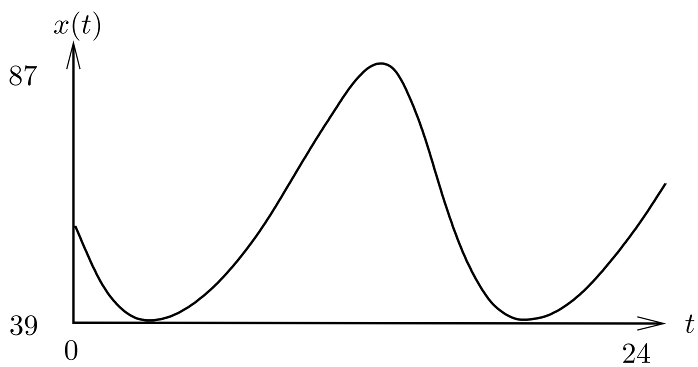

Figure 27. Scene Plot of $x(t)$ Rabbits.

An initial rabbit population of 60 and fox population of 100 causes the rabbit population $x(t)$ to fluctuate from 39 to 87. The plot uses nonlinear equations (5) with $x(0) = 60$, $y(0) = 100$.

Pesticides, Aphids and Ladybugs

The classical predator-prey equations apply for prey *Aphid* $x(t)$ and predator *Ladybug* $y(t)$, which for simplicity are assumed to be

$$\begin{array}{rcl} x'(t) & = & (1-y(t))x(t), \\ y'(t) & = & (x(t)-1)y(t), \end{array} \tag{6}$$

with units in millions.

Consider employment of an indiscriminate pesticide which kills a certain percentage of each insect. Typically available pesticide strengths are $s = 0.5$, $s = 0.75$. Strength $s = 0$ is no pesticide. We will assume hereafter that $0 \le s < 1$. The predator-prey equations mutate by adding terms for pesticide-caused death rates, resulting in the **Pesticide Model**

$$\begin{array}{rcl} x'(t) & = & (1-y(t))x(t) - s\,x(t), \\ y'(t) & = & (x(t)-1)y(t) - s\,y(t). \end{array} \tag{7}$$

Explained below in Figures 28, 29 and 30 are the results in the following table.

Table 4. Effects of Pesticide on Aphids and Ladybugs

The aphids increase and the ladybugs decrease.
The insecticide had a counterproductive effect. Aphid damage to the garden plants increased by using a pesticide.

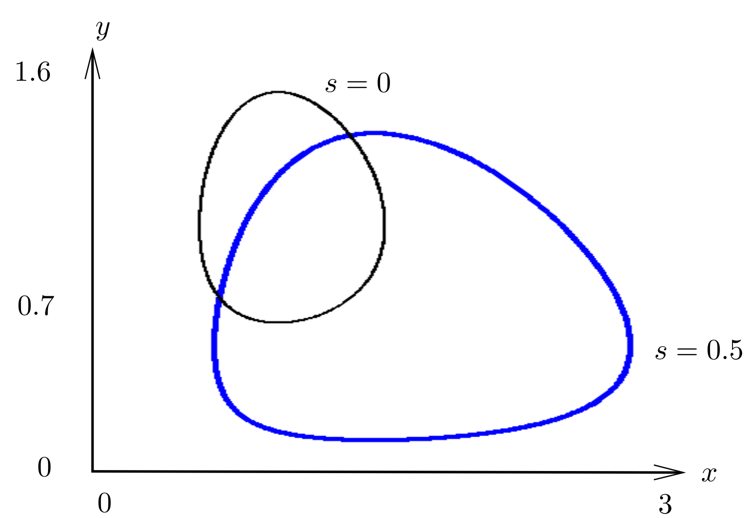

Figure 28. Aphid-Ladybug Portraits $s = 0$, $s = 0.5$.

Aphid population max and min are measured by the orbit width. Ladybug population max and min are measured by the orbit height. Both orbits use $x(0) = y(0) = 0.7$. Details appear in the x and y scene plots, *infra*.

Pesticide model (7) is equivalent to the classical predator-prey system (1) with replacements $a = 1 - s$, $b = 1 + s$. The nonlinear phase portrait for the pesticide model has according to predator-prey theory a saddle at $(0, 0)$ and a center at $(1 + s, 1 - s)$.

The scene plots in Figures 29 and 30 show that the aphids increase and the ladybugs decrease, for the two populations, $x(t)$ **aphids**, $y(t)$ **ladybugs** in pesticide system (7), with pesticide strengths $s = 0$ and $s = 0.5$ and initial populations $x(0) = 0.7$, $y(0) = 0.7$ (in millions).

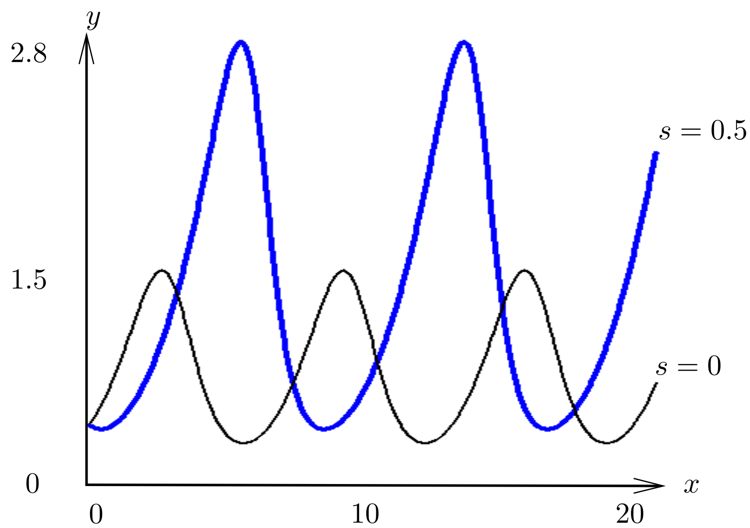

Figure 29. Aphid Scene $x(t)$**.**
Aphids increase when pesticide strength $s = 0.5$ is applied.

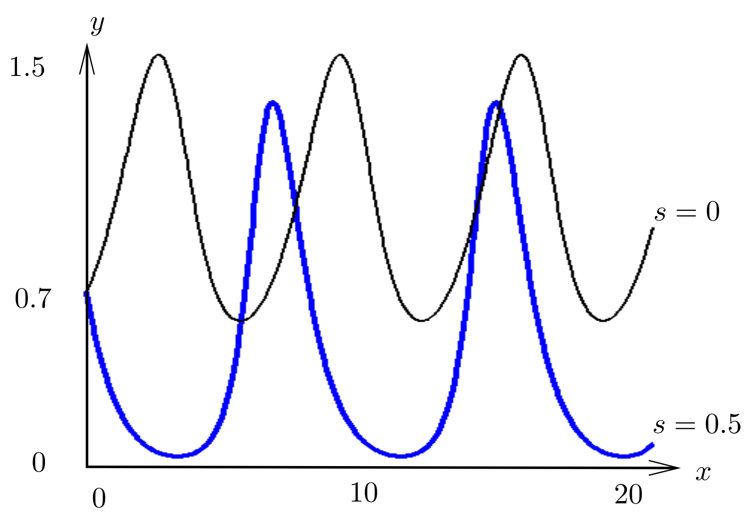

Figure 30. Ladybug Scene $y(t)$**.**
Ladybugs decrease when pesticide strength $s = 0.5$ is applied.

Competition Models

Two populations **1** and **2** feed on some constantly available food supply, e.g., two kinds of insects feed on fallen fruit. The following biological assumptions apply to model a two-population competition system.

Verhulst model **1**	Population **1** grows or decays according to the logistic equation $x'(t) = (a - bx(t))x(t)$, in the absence of population **2**.
Verhulst model **2**	Population **2** grows or decays according to the logistic equation $y'(t) = (c - dy(t))y(t)$, in the absence of population **1**.
Chance encounters	Population **1** decays at a rate $-pxy$, $p > 0$, due to chance encounters with population **2**. Population **2** decays at a rate $-qxy$, $q > 0$, due to chance encounters with population **1**.

Adding the Verhulst rates and the chance encounter rates gives the **Volterra competition system**

$$(8) \qquad \begin{aligned} x'(t) &= (a - bx(t) - py(t))x(t), \\ y'(t) &= (c - dy(t) - qx(t))y(t). \end{aligned}$$

The equations show that each population satisfies a time-varying first order differential equation $u'(t) = r(t)u(t)$ in which the rate function $r(t)$ depends on time. For initial population sizes near zero, the two differential equations essentially reduce to the Malthusian growth models $x'(t) = ax(t)$ and $y'(t) = cy(t)$. As viewed from Malthus' law $u' = ru$, population **1** has growth rate $r = a - bx - py$ which decreases if population **2** grows, resulting in a reduction of population **1**. Likewise, population **2** has growth rate $r = c - dy - qx$, which reduces population **2** as population **1** grows. While a, c are Malthusian growth rates, constants b, d measure **inhibition** (due to lack of food or space) and constants p, q measure **competition**.

Equilibria

The equilibrium points $\vec{u}$ satisfy $\vec{F}(\vec{u}) = \vec{0}$ where $\vec{F}$ is defined by

$$(9) \qquad \vec{F}(\vec{u}) = \begin{pmatrix} (a - bx - py)x \\ (c - dy - qx)y \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To isolate the most important applications, the assumption will be made of exactly four roots in population quadrant I. This is equivalent to the condition $bd - qp \neq 0$ plus all equilibria have nonnegative coordinates.

Three of the four equilibria are found to be $(0,0)$, $(a/b,0)$, $(0,c/d)$. The last two represent the carrying capacities of the Verhulst models in the absence of the second population. The fourth equilibrium (x_0,y_0) is found as the *unique root* $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ of the linear system

$$\begin{pmatrix} b & p \\ q & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

which according to Cramer's rule is

$$x_0 = \frac{ad-pc}{bd-qp}, \quad y_0 = \frac{bc-qa}{bd-qp}.$$

Linearized Competition System

The Jacobian matrix $J(x,y)$ is computed from the partial derivatives of system variables f, g, which are found as follows.

$$\begin{array}{lcccr} f(x,y) & = & (a - b\,x - p\,y)\,x, & = & a\,x - b\,x^2 - p\,xy \\ g(x,y) & = & (c - d\,y - q\,x)\,y & = & c\,y - d\,y^2 - q\,xy \\ f_x & = & \frac{\partial}{\partial x}(a\,x - b\,x^2 - p\,xy) & = & a - 2b\,x - p\,y \\ f_y & = & \frac{\partial}{\partial y}(a\,x - b\,x^2 - p\,xy) & = & -p\,x \\ g_x & = & \frac{\partial}{\partial x}(c\,y - d\,y^2 - q\,xy) & = & -q\,y \\ g_y & = & \frac{\partial}{\partial y}(c\,y - d\,y^2 - q\,xy) & = & c - 2d\,y - q\,x \end{array}$$

The Jacobian matrix is given explicitly by

$$(10) \quad J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a-2bx-py & -px \\ -qy & c-2dy-qx \end{pmatrix}.$$

The matrix J is evaluated at an equilibrium point (a root of $\vec{F}(\vec{u}) = \vec{0}$) to obtain a 2×2 matrix A for the linearized system $\frac{d}{dt}\vec{v}(t) = A\,\vec{v}(t)$. The four linearized systems are:

Equilibrium $(0,0)$
Nodal Repeller
$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\vec{u}(t)$$

Equilibrium $(a/b,0)$
Saddle or Nodal Attractor
$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} -a & -ap/b \\ 0 & c-qa/b \end{pmatrix}\vec{u}(t)$$

Equilibrium $(0,c/d)$
Saddle or Nodal Attractor
$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} a-cp/d & 0 \\ -qc/d & -c \end{pmatrix}\vec{u}(t)$$

Equilibrium (x_0,y_0)
Saddle or Nodal Attractor
$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} -bx_0 & -px_0 \\ -qy_0 & -dy_0 \end{pmatrix}\vec{u}(t)$$

Equilibria $(a/b, 0)$ and $(0, c/d)$ are either both saddles or both nodal atractors, accordingly as $bd - qp > 0$ or $bd - qp < 0$, because of the requirement that a, b, c, d, p, q, x_0, y_0 be positive.

The analysis of equilibrium (x_0, y_0) is made by computing the eigenvalues λ of the linearized system, from characteristic equation $\lambda^2 + (bx_0 + dy_0)\lambda + (bd - pq)x_0y_0 = 0$, giving

$$\lambda = \frac{1}{2}\left(-(bx_0 + dy_0) \pm \sqrt{D}\right), \quad \text{where} \quad D = (bx_0 - dy_0)^2 + 4pqx_0y_0.$$

Because $D > 0$, the equilibrium is a saddle when the roots have opposite sign, and it is a nodal attractor when both roots are negative. The saddle case is $D > (bx_0 + dy_0)^2$ or equivalently $4x_0y_0(pq - bd) > 0$, which reduces to $bd - qp < 0$. In summary:

> If $bd - qp > 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, (x_0, y_0) are respectively a saddle, saddle, nodal attractor.
>
> If $bd - qp < 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, (x_0, y_0) are respectively a nodal attractor, nodal attractor, saddle.

Biological Meaning of $bd - qp$ Negative or Positive

The quantities bd and qp are measures of inhibition and competition.

Survival-Extinction	The inequality $bd - qp < 0$ means that competition qp is large compared with inhibition bd. The equilibrium point (x_0, y_0) is unstable in this case, which biologically means that the two species cannot coexist: **survival** for one species and **extinction** for the other species.
Co-existence	The inequality $bd - qp > 0$ means that competition qp is small compared with inhibition bd. The equilibrium point (x_0, y_0) is asymptotically stable in this case, which biologically means the two species **co-exist**.

Survival of One Species

Consider populations $x(t)$ and $y(t)$ that satisfy the competition model

$$(11) \qquad \begin{aligned} x'(t) &= x(t)(24 - x(t) - 2y(t)), \\ y'(t) &= y(t)(30 - y(t) - 2x(t)). \end{aligned}$$

We apply the general competition theory with $a = 24$, $b = 1$, $p = 2$, $c = 30$, $d = 1$, $q = 2$. The equilibrium points are $(0, 0)$, $(0, 30)$, $(24, 0)$, $(12, 6)$, shown in Figure 31 as solid circles and squares. Eigenvalues are computed from Jacobian matrix $J(x,y) = \begin{pmatrix} 24-2x-2y & -2x \\ -2y & 30-2y-2x \end{pmatrix}$ evaluated at the four equilibria. The answers:

Equilibrium $(0,0)$**:** $\lambda = 24, 30$, nodal repeller.

Equilibrium $(0,30)$**:** $\lambda = -36, -30$, nodal attractor.

Equilibrium $(24,0)$**:** $\lambda = -24, -18$, nodal attractor.

Equilibrium $(12,6)$**:** $\lambda = 8.23, -26.23$, saddle.

The **Paste Theorem** says that the linear portraits can be pasted atop the four equilibria in the nonlinear phase portrait. The tuned portrait appears in Figure 31, clipped to the population quadrant $x \geq 0, y \geq 0$.

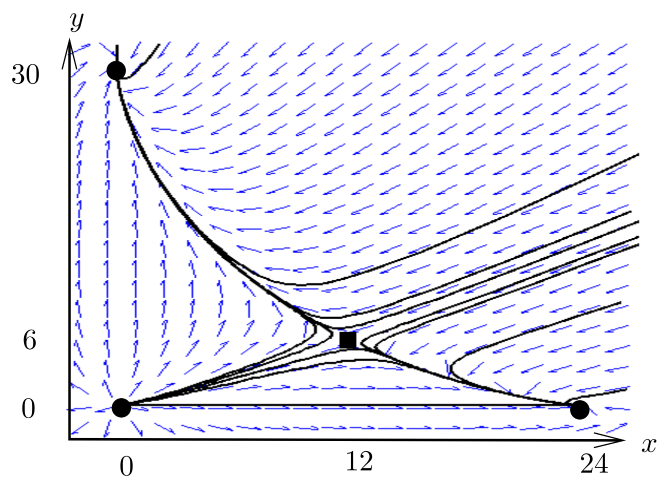

Figure 31. Survival of One Species.

Portrait for system (11). Equilibria are $(0,0)$, $(0,30)$, $(24,0)$ and $(12,6)$, classified respectively as nodal repeller, nodal attractor, nodal attractor and saddle. The population with initial advantage survives, while the other dies out.

Co-existence

Consider populations $x(t)$ and $y(t)$ that satisfy the competition model

$$(12) \qquad \begin{aligned} x'(t) &= x(t)(24 - 2x(t) - y(t)), \\ y'(t) &= y(t)(30 - 2y(t) - x(t)). \end{aligned}$$

We apply the general competition theory with $a = 24$, $b = 2$, $p = 1$, $c = 30$, $d = 2$, $q = 1$. The equilibrium points are $(0,0)$, $(0,15)$, $(12,0)$ and $(6,12)$, shown in Figure 32 as solid circles and squares. Eigenvalues are computed from Jacobian matrix $J(x,y) = \begin{pmatrix} 24-4x-y & -x \\ -y & 30-4y-x \end{pmatrix}$ evaluated at the four equilibria. The answers:

Equilibrium $(0,0)$**:** $\lambda = 24, 30$, nodal repeller.

Equilibrium $(0,30)$**:** $\lambda = 18, -24$, saddle.

Equilibrium $(24,0)$**:** $\lambda = 9, -30$, saddle.

Equilibrium $(12,6)$**:** $\lambda = -7.61, -28.39$, nodal attractor.

The linear portraits can be pasted atop the four equilibria in the nonlinear phase portrait, according to the **Paste Theorem**. Figure 32 is the tuned portrait.

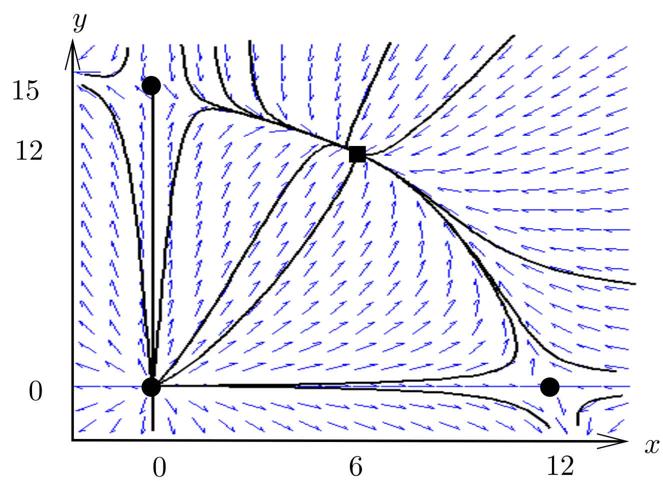

Figure 32. Coexistence.

Phase portrait of system (12). The equilibria are $(0,0)$, $(0,15)$, $(12,0)$ and $(6,12)$, classified respectively as nodal repeller, saddle, saddle, nodal attractor. A solution with $x(0) > 0$, $y(0) > 0$ limits at $t = \infty$ to the solid square $(6,12)$. **Coexistence states** are $x = 6, y = 12$.

Alligators, Explosion and Extinction

Let us assume a competition-type model (8) in which the Verhulst dynamics has explosion-extinction type. Thus, we take the signs of a, b, c, d in (8) to be negative, but p, q are still positive. The populations $x(t)$ and $y(t)$ are unsophisticated in the sense that each population in the absence of the other is subject to only the possibilities of population explosion or population extinction.

It can be verified for this general setting, although we shall not attempt to do so here, that the population quadrant $x(0) > 0$, $y(0) > 0$ is separated into two regions I and II, whose common boundary is a separatrix consisting of three equilibria and two orbits. An orbit starting in region I will have (a) $x(\infty) = 0$, $y(\infty) = \infty$, or (b) $x(\infty) = \infty$, $y(\infty) = 0$, or

(c) $x(\infty) = \infty$, $y(\infty) = \infty$. Orbits starting in region II will satisfy (d) $x(\infty) = 0$, $y(\infty) = 0$. The biological conclusion is that either population explosion or extinction occurs for each population.

Consider the instance

$$(13) \qquad \begin{aligned} x'(t) &= x(t)(x(t) - y(t) - 4), \\ y'(t) &= y(t)(x(t) + y(t) - 8). \end{aligned}$$

We will apply the general competition theory with $a = 24$, $b = 2$, $p = 1$, $c = 30$, $d = 2$, $q = 1$. The equilibria are $(0, 0)$, $(0, 8)$, $(4, 0)$ and $(6, 2)$, shown in Figure 33 as solid circles and a square. Eigenvalues λ are computed from Jacobian matrix $J(x, y) = \begin{pmatrix} 2x - y - 4 & -x \\ -y & x + 2y - 8 \end{pmatrix}$ evaluated at the four equilibria. The answers:

Equilibrium $(0, 0)$**:** $\lambda = -4, -8$, nodal attractor.

Equilibrium $(0, 30)$**:** $\lambda = 8, -12$, saddle.

Equilibrium $(24, 0)$**:** $\lambda = 4, -4$, saddle.

Equilibrium $(12, 6)$**:** $\lambda = 4 \pm 2.83\,i$, spiral repeller.

The **Paste Theorem** predicts the tuned portrait in Figure 33.

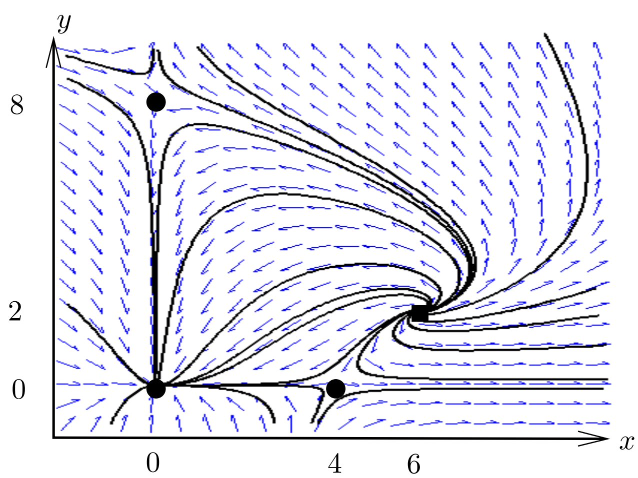

Figure 33. Population Explosion or Extinction.

Phase portrait of system (13). The equilibria are $(0, 0)$, $(0, 8)$, $(4, 0)$ and $(6, 2)$, classified respectively as nodal attractor, saddle, saddle and spiral repeller. The node and two saddles are marked with a solid disk and the spiral repeller is marked with a solid square.

Exercises 10.4

Predator-Prey Models.

Consider the system

$$\begin{aligned} x'(t) &= \frac{1}{250}(1-2y(t))x(t), \\ y'(t) &= \frac{3}{500}(2x(t)-1)y(t). \end{aligned}$$

1. (System Variables) The system has vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

2. (System Parameters) Identify the values of a, b, c, d, p, q, as used in the textbook's predator-prey system.

3. (Identify Predator and Prey) Which of $x(t), y(t)$ is the predator?

4. (Switching Predator and Prey) Give an example of a predator-prey system in which $x(t)$ is the predator and $y(t)$ is the prey.

Implicit Solution Predator-Prey.

These exercises prove the equation

$$a\,\ln|y| + b\,\ln|x| - q\,x - p\,y = C.$$

5. (First Order Equation) Verify from the chain rule of calculus the first order equation

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y}{x}\frac{qx-b}{a-py}.$$

6. (Separated Variables) Verify

$$\left(\frac{a}{y} - p\right) dy = \left(q - \frac{b}{x}\right) dx.$$

7. (Quadrature) Integrate the equation of the previous exercise to obtain

$$a\,\ln|y| - p\,y = q\,x - b\,\ln|x| = C.$$

Then re-arrange to obtain the reported implicit solution.

8. (Energy Function) Define $E(t) = a\ln|u| - pu$. Show that $dE/du = (a-pu)/u$. Then show that $dE/du < 0$ for $a > 0$, $p > 0$ and $a/p < u < \infty$.

Linearized Predator-Prey System.

Consider

$$\begin{aligned} x'(t) &= (100-2y(t))x(t), \\ y'(t) &= (2x(t)-160)y(t). \end{aligned}$$

9. (Find Equilibria) Verify equilibria $(0,0), (80,50)$.

10. (Jacobian Matrix) Compute $J(x,y)$ for each x, y. Then find $J(0,0)$ and $J(80,50)$.

11. (Transit Time) Find the trip time of an orbit about $(0,0)$ for system $\frac{d}{dt}\vec{v} = \begin{pmatrix} 0 & -160 \\ 100 & 0 \end{pmatrix}\vec{v}$, the linearization about $(80,50)$.

12. (Paste Theorem) Describe the local figures expected near equilibria in the nonlinear phase portrait.

Rabbits and Foxes.

Consider

$$\begin{aligned} x'(t) &= \frac{1}{200}\,x(t)(50-y(t)), \\ y'(t) &= \frac{1}{100}\,y(t)(x(t)-40). \end{aligned}$$

13. (Equilibria) Verify equilibria $(0,0)$, $(40,50)$, showing all details.

14. (Jacobian) Compute Jacobian $J(x,y)$, then $J(0,0)$ and $J(40,50)$.

15. (Rabbit Oscillation) The linear and nonlinear scenes for $x(t)$ must approximate each other. Find estimates for the max and min of rabbits and their period of oscillation, for the nonlinear system.

Pesticides.

Consider the system

$$\begin{aligned} x'(t) &= (10-y(t))x(t) - s_1x(t), \\ y'(t) &= (x(t)-20)y(t) - s_2y(t). \end{aligned}$$

16. (Equilibria) Show details for computing the pesticide system equilibria $(0,0)$, $(20+s_2, 10-s_1)$, where s_1, s_2 are the pesticide death rates.

17. (Average Populations) Explain: A field biologist should count, on the average, populations of about $20+s_2$ prey and $10-s_1$ predators.

Survival of One Species. Consider

$$\begin{aligned} x'(t) &= x(t)(24-x(t)-2y(t)), \\ y'(t) &= y(t)(30-y(t)-2x(t)). \end{aligned}$$

18. (Interactions) Show that doubling either x or y causes the interaction term $2xy$ to double.

19. (Equilibria) Find all equilibria.

20. (Linearization) Find the linearized systems $\frac{d}{dt}\vec{v} = J(x_0, y_0)\vec{v}$ for each equilibrium point (x_0, y_0).

21. (Nonlinear Classification) Classify each equilibrium point (x_0, y_0) as center, spiral, node, saddle, using the **Paste Theorem**. Make a phase portrait which confirms the classifications.

Co-existence.

Explosion and Extinction.

10.5 Mechanical Models

Nonlinear Spring-Mass System

The classical linear undamped spring-mass system is modeled by the equation $mx''(t)+kx(t)=0$. This equation describes the excursion $x(t)$ from equilibrium $x=0$ of a mass m attached to a spring of Hooke's constant k, with no damping and no external forces.

In the nonlinear theory, the Hooke's force term $-kx$ is replaced by a **restoring force** $F(x)$ which satisfies these four requirements:

Equilibrium 0**.** The equation $F(0)=0$ is assumed, which gives $x=0$ the status of a rest position.

Oddness. The equation $F(-x)=-F(x)$ is assumed, which says that the force F depends only upon the magnitude of the excursion from equilibrium, and not upon its direction. Then force F acts to **restore** the mass to its equilibrium position, like a Hooke's force $x\to kx$.

Zero damping. The damping effects always present in a real physical system are ignored. In linear approximations, it would be usual to assume a viscous damping effect $-cx'(t)$; from this viewpoint we assume $c=0$.

Zero external force. There is no external force acting on the system. In short, only two forces act on the mass, (1) Newton's second law and (2) restoring force F.

The competition method applies to model the nonlinear spring-mass system via the two competing forces $mx''(t)$ and $F(x(t))$. The dynamical equation:

$$mx''(t)+F(x(t))=0. \tag{1}$$

Soft and Hard Springs

A restoring force F modeled upon Hooke's law is given by the equation $F(x)=kx$. With this force, the nonlinear spring-mass equation (1) becomes the undamped linear spring-mass system

$$mx''(t)+kx(t)=0. \tag{2}$$

The linear equation can be thought to originate by replacing the actual spring force F by the first nonzero term of its Taylor series

$$F(x)=F(0)+F'(0)x+F''(0)\frac{x^2}{2!}+\cdots.$$

The assumptions $F(-x) = -F(x)$ and $F(0) = 0$ imply that $F(x)$ is a function of the form $F(x) = xG(x^2)$, hence all even terms in the Taylor series of F are zero.

Linear approximations to the force F drop the quadratic terms and higher from the Taylor series. More accurate nonlinear approximations are obtained by retaining extra Taylor series terms.

A restoring force F is called **hard** or **soft** provided it is given by a truncated Taylor series as follows.

Hard spring	$F(x) = kx + \beta x^3$, $\beta > 0$.
Soft spring	$F(x) = kx - \beta x^3$, $\beta > 0$.

For small excursions from equilibrium $x = 0$, a hard or soft spring force has magnitude approximately the same as the linear Hooke's force $F(x) = kx$.

Energy Conservation. Each solution $x(t)$ of the nonlinear spring-mass equation $mx''(t) + F(x(t)) = 0$ satisfies on its domain of existence the **conservation law**

$$(3) \qquad \frac{m}{2}(x'(t))^2 + \int_{x(0)}^{x(t)} F(u)\,du = C, \quad C \equiv \frac{m}{2}(x'(0))^2.$$

To prove the law, multiply the nonlinear differential equation by $x'(t)$ to obtain $mx''(t)x'(t) + F(x(t))x'(t) = 0$, then apply quadrature to obtain (3).

Kinetic and Potential Energy. Using $v = x'(t)$, the term $mv^2/2$ in (3) is called the **kinetic energy** (KE) and the term $\int_{x_0}^{x} F(u)du$ is called the **potential energy** (PE). Equation (3) says that $KE + PE = C$ or that *energy is constant* along trajectories.

The conservation laws for the soft and hard nonlinear spring-mass systems, using position-velocity notation $x = x(t)$ and $y = x'(t)$, are therefore given by the equations

$$(4) \qquad my^2 + kx^2 + \frac{1}{2}\beta x^4 = C_1, \quad C_1 = \text{constant} > 0,$$

$$(5) \qquad my^2 + kx^2 - \frac{1}{2}\beta x^4 = C_2, \quad C_2 = \text{constant}.$$

Phase Plane and Scenes. Nonlinear behavior is commonly graphed in the **phase plane**, in which $x = x(t)$ and $y = x'(t)$ are the position and velocity of the mechanical system. The plots of t versus $x(t)$ or $x'(t)$ are called **scenes**; these plots are invaluable for verifying periodic behavior and stability properties.

Hard spring

The only equilibrium for a hard spring $x' = y$, $my' = -kx - \beta x^3$ is the origin $x = y = 0$. Conservation law (4) describes a closed curve in the phase plane, which implies that trajectories are periodic orbits that encircle the equilibrium point $(0, 0)$. The classification of **center** applies. See Figures 34 and 35.

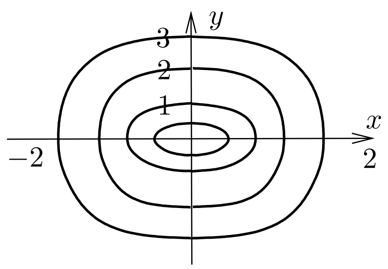

Figure 34. Hard spring
$x''(t) + x(t) + 2x^3(t) = 0$.
Phase portrait for $x' = y$,
$y' = -2x^3 - x$ on $|x| \le 2$, $|y| \le 3.5$.
Initial data: $x(0) = 0$ and $y(0) = 1/2$, 1, 2, 3.

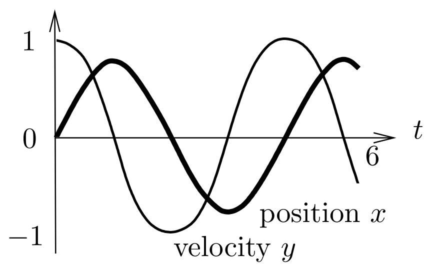

Figure 35. Hard spring
$x''(t) + x(t) + 2x^3(t) = 0$.
Coordinate scenes for $x' = y$,
$y' = -2x^3 - x$, $x(0) = 0$, $y(0) = 1$.

More intuition about the orbits can be obtained by finding the energy C_1 for each orbit. The value of C_1 decreases to zero as orbits close down upon the origin. Otherwise stated, the xyz-plot with $z = C_1$ has a minimum at the origin, which physically means that the equilibrium state $x = y = 0$ minimizes the energy. See Figure 36.

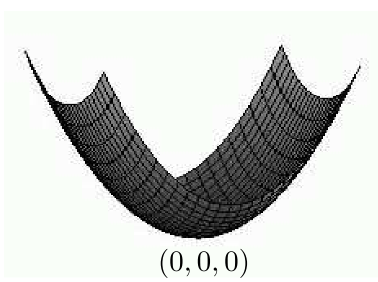

Figure 36. Hard spring energy minimization.
Plot for $x''(t) + x(t) + 2x^3(t) = 0$, using $z = y^2 + x^2 + x^4$ on $|x| \le 1/2$, $|y| \le 1$. The minimum is realized at $x = y = 0$.

Soft Spring

There are three equilibria for a soft spring

$$\begin{aligned} x' &= y, \\ my' &= -kx + \beta x^3. \end{aligned}$$

They are $(-\alpha, 0)$, $(0, 0)$, $(\alpha, 0)$, where $\alpha = \sqrt{k/\beta}$. If $(x(0), y(0))$ is given not at these points, then the mass undergoes motion. In short, the stationary mass positions are at the equilibria.

Linearization at the equilibria reveals part of the phase portrait. The linearized system at the origin is the system $x' = y$, $my' = -kx$, equivalent to the equation $mx'' + kx = 0$. It has a center at the origin. This implies the origin for the soft spring is either a center or a spiral. The other two equilibria have linearized systems equivalent to the equation $mx'' - 2kx = 0$; they are saddles.

The phase plot in Figure 37 shows separatrices, which are unions of solution curves and equilibrium points. Orbits in the phase plane, on either side of a separatrix, have physically different behavior. Shown is a center behavior interior to the union of the separatrices, while outside all orbits are unbounded.

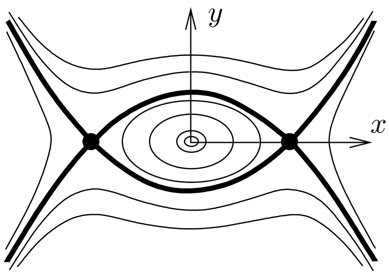

Figure 37. Soft spring $x''(t) + x(t) - 2x^3(t) = 0$**.** A phase portrait for $x' = y$, $y' = 2x^3 - x$ on $|x| \le 1.2$, $|y| \le 1.2$. The 8 separatrices are the 6 bold curves plus the two equilibria $(\sqrt{0.5}, 0)$, $(-\sqrt{0.5}, 0)$.

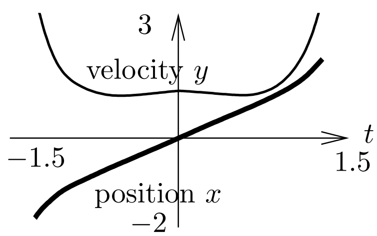

Figure 38. Soft spring $x''(t) + x(t) - 2x^3(t) = 0$**.** Coordinate scenes for $x' = y$, $y' = 2x^3 - x$, $x(0) = 0$, $y(0) = 4$.

Nonlinear Pendulum

Consider a nonlinear undamped pendulum of length L making angle $\theta(t)$ with the gravity vector. The **nonlinear pendulum equation** is given by

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0 \tag{6}$$

and its linearization at $\theta = 0$, called the **linearized pendulum equation**, is

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\theta(t) = 0. \tag{7}$$

The linearized equation is valid only for small values of $\theta(t)$, because of the assumption $\sin\theta \approx \theta$ used to obtain (7) from (6).

Damped Pendulum

Physical pendulums are subject to friction forces, which we shall assume proportional to the velocity of the pendulum. The corresponding model which includes frictional forces is called the **damped pendulum equation**:

(8) $$\frac{d^2\theta(t)}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta(t)) = 0.$$

It can be written as a first order system by setting $x(t) = \theta(t)$ and $y(t) = \theta'(t)$:

(9) $$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\sin(x(t)) - cy(t). \end{aligned}$$

Undamped Pendulum

The position-velocity differential equations for the undamped pendulum are obtained by setting $x(t) = \theta(t)$ and $y(t) = \theta'(t)$:

(10) $$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\sin(x(t)). \end{aligned}$$

Equilibrium points of nonlinear system (10) are at $y = 0$, $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ with corresponding linearized system (see the exercises)

(11) $$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\cos(n\pi)x(t). \end{aligned}$$

The characteristic equation of linear system (11) is $r^2 - g/L(-1)^n = 0$, because $\cos(n\pi) = (-1)^n$. The roots have different character depending on whether or not n is odd or even.

Even $n = 2m$. Then $r^2 + g/L = 0$ and the linearized system (11) is a **center**. The orbits of (11) are concentric circles surrounding $x = n\pi$, $y = 0$.

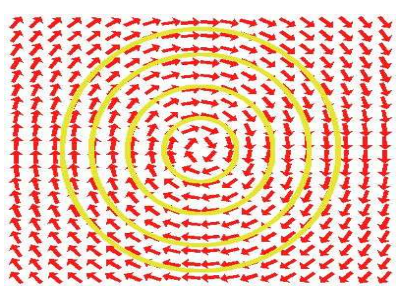

Figure 39. Linearized pendulum at $x = 2m\pi$, $y = 0$.
Orbits are concentric circles.

Odd $n = 2m + 1$. Then $r^2 - g/L = 0$ and the linearized system (11) is a **saddle**. The orbits of (11) are hyperbolas with center $x = n\pi$, $y = 0$.

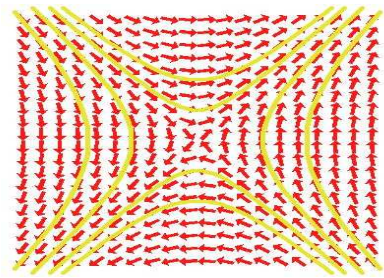

Figure 40. Linearized pendulum at $x = (2m+1)\pi$, $y = 0$.
Orbits are hyperbolas.

Drawing the Nonlinear Phase Diagram. The idea of the plot is to copy the linearized diagram onto the local region centered at the equilibrium point, when possible. The copying is guaranteed to be correct for the saddle case, but a center must be copied either as a spiral or a center. We must do extra analysis to determine the figure to copy in the case of the center.

The orbits trace an xy-curve given by integrating the separable equation

$$\frac{dy}{dx} = \frac{-g}{L}\frac{\sin x}{y}.$$

Then the conservation law for the mechanical system is

$$\frac{1}{2}y^2 + \frac{g}{L}(1 - \cos x) = E$$

where E is a constant of integration. This equation is arranged so that E is the sum of the kinetic energy $y^2/2$ and the potential energy $g(1 - \cos x)/L$, therefore E is the total mechanical energy. Using the double angle identity $\cos 2\phi = 1 - 2\sin^2\phi$ the conservation law can be written in the shorter form

$$y^2 + \frac{4g}{L}\sin^2(x/2) = 2E$$

When the energy E is small, $E < 2g/L$, then the pendulum never reaches the vertical position and it undergoes sustained periodic oscillation: the stable equilibria $(0, 2k\pi)$ have a local center structure.

When the energy E is large, $E > 2g/L$, then the pendulum reaches the vertical position and goes over the top repeatedly, represented by a saddle structure. The statement is verified from the two explicit solutions $y = \pm\sqrt{2E - 4g\sin^2(x/2)/L}$.

The energy equation $E = 2g/L$ produces the separatrices, which consist of equilibrium points plus solution curves which limit to the equilibria as $t \to \pm\infty$.

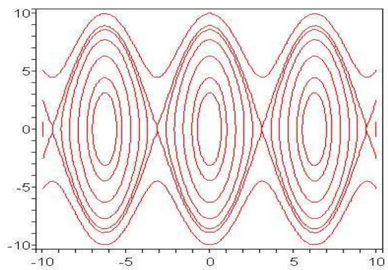

Figure 41. Nonlinear pendulum phase diagram.
Centers at $(-2\pi, 0)$, $(0, 0)$, $(2\pi, 0)$. Saddles at $(-3\pi, 0)$, $(-pi, 0)$, $(\pi, 0)$, $(3\pi, 0)$. Separatrices are generated from equilibria and $G(x, y) = 2E$, with $E = 2g/L$ and $g/L = 10$.