Classification of Phase Portraits at Equilibria for $\vec{u}'(t) = \vec{f}(\vec{u}(t))$

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Transfer of Local Linearized Phase Portrait

THEOREM.
Let $\vec{\mathbf{u}}_0$ be an equilibrium point of the nonlinear dynamical system

$$\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t)).$$

Assume the Jacobian of $\vec{\mathbf{f}}(\vec{\mathbf{u}})$ at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ is matrix A and $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$ has linear classification **saddle**, **node**, **center** or **spiral** at its equilibrium point $(0, 0)$.

Then the nonlinear system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ at equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ has the same classification, with the following exceptions:

> If the linear classification at $(0, 0)$ for $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$ is a node or a center, then the nonlinear classification at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ might be a spiral.

The exceptions in terms of roots of the characteristic equation: $\lambda_1 = \lambda_2$ (real equal roots) and $\lambda_1 = \overline{\lambda_2} = bi$ ($b > 0$, purely complex roots).

Transfer of Local Linearized Stability

THEOREM.
Let $\vec{\mathbf{u}}_0$ be an equilibrium point of the nonlinear dynamical system

$$\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t)).$$

Assume the Jacobian of $\vec{\mathbf{f}}(\vec{\mathbf{u}})$ at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ is matrix A. Then the nonlinear system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ has the same stability as $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$ with the following exception:

> If the linear classification at $(0, 0)$ for $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$ is a center, then the nonlinear classification at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ might be either stable or unstable.

How to Classify Linear Equilibria

- Assume the linear system is $\mathbf{2} \times \mathbf{2}$, $\vec{\mathbf{u}}' = \boldsymbol{A}\vec{\mathbf{u}}$.
- Compute the roots $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ of the characteristic equation of $\boldsymbol{A}$.
- Find the atoms $\boldsymbol{A}_1(\boldsymbol{t})$, $\boldsymbol{A}_2(\boldsymbol{t})$ for these two roots.
- If the atoms have sine and cosine factors, then a rotation is implied and the classification is either a **center** or **spiral**. Pure harmonic atoms [no exponentials] imply a center, otherwise it's a spiral.
- If the atoms are exponentials, then the classification is a non-rotation, a **node** or **saddle**. Take limits of the atoms at $\boldsymbol{t} = \infty$ and also $\boldsymbol{t} = -\infty$. If one limit answer is $\boldsymbol{A}_1 = \boldsymbol{A}_2 = \mathbf{0}$, then it's a node, otherwise it's a saddle.

Justification of the Classification Method

The Cayley-Hamilton-Ziebur theorem implies that the general solution of

$$\vec{\mathbf{u}}' = \boldsymbol{A}\vec{\mathbf{u}}$$

is the equation

$$\vec{\mathbf{u}}(\boldsymbol{t}) = \boldsymbol{A}_1(\boldsymbol{t})\vec{\boldsymbol{d}}_1 + \boldsymbol{A}_2(\boldsymbol{t})\vec{\mathbf{d}}_2$$

where $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are the atoms corresponding to the roots $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ of the characteristic equation of $\boldsymbol{A}$. Although $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$ are not arbitrary, the classification only depends on the roots and hence only on the atoms. We construct examples of the behavior by choosing $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, for example,

$$\vec{\mathbf{d}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{d}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If the atoms were $\cos \boldsymbol{t}$, $\sin \boldsymbol{t}$, then the solution by C-H-Z would be $\boldsymbol{x} = \cos \boldsymbol{t}$, $\boldsymbol{y} = \sin \boldsymbol{t}$. Analysis of the trajectory shows a circle, hence we expect a **center** at $(0, 0)$. Similar examples can be invented for the other cases of a **spiral**, **saddle**, or **node**, by considering possible pairs of atoms.

Three Examples

Consider the nonlinear systems and selected equilibrium points. The third example has infinitely many equilibria.

Spiral–Saddle $\begin{cases} x' = x + y, \\ y' = 1 - x^2. \end{cases}$ Equilibria $(1, -1)$, $(-1, 1)$

Center–Saddle $\begin{cases} x' = y, \\ y' = -20x + 5x^3. \end{cases}$ Equilibria $(0, 0)$, $(2, 0)$, $(-2, 0)$

Node–Saddle $\begin{cases} x' = 3\sin(x) + y, \\ y' = \sin(x) + 2y. \end{cases}$ Equilibria $(2\pi, 0)$, $(\pi, 0)$

Spiral-saddle Example

The nonlinear function and Jacobian are

$$\vec{\mathbf{f}}(x,y) = \begin{pmatrix} x+y \\ 1-x^2 \end{pmatrix}, \quad A(x,y) = \begin{pmatrix} 1 & 1 \\ -2x & 0 \end{pmatrix}.$$

Then $A(1,-1) = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$ and $A(-1,1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$.

- The characteristic equations are $\lambda^2 - \lambda + 2 = 0$ and $\lambda^2 - \lambda - 2 = 0$ with roots $\frac{1}{2} \pm \frac{1}{2}\sqrt{7}i$ and $2, -1$, respectively.
- The atoms for $A(1,-1)$ are $e^{t/2}\cos(\sqrt{7}t/2)$, $e^{t/2}\sin(\sqrt{7}t/2)$. Rotation implies a center or spiral. No pure harmonics, so it's a spiral. The limit at $t = -\infty$ is zero for both atoms, so it's stable at minus infinity, implying unstable at infinity.
- The atoms for $A(-1,1)$ are e^{2t}, e^{-t}. No rotation implies a node or saddle. Neither the limit at infinity nor at minus infinity gives zero, so it's a saddle.

Center-saddle Example

The nonlinear function and Jacobian are

$$\vec{\mathbf{f}}(x,y) = \begin{pmatrix} y \\ -20x + 5x^3 \end{pmatrix}, \quad A(x,y) = \begin{pmatrix} 0 & 1 \\ -20 + 15x^2 & 0 \end{pmatrix}.$$

Then $A(0,0) = \begin{pmatrix} 0 & 1 \\ -20 & 0 \end{pmatrix}$ and $A(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ 40 & 0 \end{pmatrix}$.

- The characteristic equations are $\lambda^2 + 20 = 0$ and $\lambda^2 - 40 = 0$ with roots $\pm\sqrt{20}i$ and $\pm\sqrt{40}$, respectively.
- The atoms for $A(0,0)$ are $\cos(\sqrt{20}t)$, $\sin(\sqrt{20}t)$. Rotation implies a center or spiral. The atoms are pure harmonics, so it's a center. The nonlinear system can be a center or a spiral and either stable or unstable. The issue is decided by a computer algebra system to be a center.
- The atoms for $A(\pm 2, 0)$ are e^{bt}, e^{-bt}, where $b = \sqrt{40}$. No rotation implies a node or saddle. Neither the limit at infinity nor at minus infinity gives zero, so it's a saddle.

Node-saddle Example

The nonlinear function and Jacobian are

$$\vec{f}(x,y) = \begin{pmatrix} 3\sin x + y \\ \sin x + 2y \end{pmatrix}, \quad A(x,y) = \begin{pmatrix} 3\cos x & 1 \\ \cos x & 2 \end{pmatrix}.$$

Then $A(2\pi, 0) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ and $A(\pi, 0) = \begin{pmatrix} -3 & 1 \\ -1 & 2 \end{pmatrix}$.

- The characteristic equations are $\lambda^2 - 5\lambda + 5 = 0$ and $\lambda^2 + \lambda - 5 = 0$ with roots $\frac{1}{2}(5 \pm \sqrt{5}) = 3.6, 1.38$ and $\frac{1}{2}(-1 \pm \sqrt{21}) = 1.79, -2.79$, respectively.
- The atoms for $A(2\pi, 0)$ are e^{at}, e^{bt} with $a > 0$, $b > 0$. No rotation implies a node or saddle. The atoms limit to zero at $t = -\infty$, so one end is stable, which eliminates the saddle. It's a node, unstable at infinity.
- The atoms for $A(\pi, 0)$ are e^{at}, e^{bt}, where $a > 0$ and $b < 0$. No rotation implies a node or saddle. Neither the limit at infinity nor at minus infinity gives zero, so it's a saddle.
- The two classifications and their stability transfers to the nonlinear system. The only case when a node does not automatically transfer is the case of equal roots.