The Basic Trial Solution Method

Outlined here is the method for a second order differential equation $ay'' + by' + cy = f(x)$. The method applies unchanged for nth order equations.

Step **1**. Repeatedly differentiate the atoms of $f(x)$ until no new atoms appear. Collect the distinct atoms so found into a list of k atoms. Multiply these atoms by **undetermined coefficients** $d_1, d_2, \ldots, d_k$, then add, defining **trial solution** y.

Step **2**. Substitute y into the differential equation.

Fixup Rule I. If some variable d_p is missing in the equation, then step 2 fails. Correct the trial solution as follows. Variable d_p appears in y as term $d_p A$, where A is an atom. Multiply A and all its related atoms B by x. The modified expression y is called a **corrected trial solution**. Repeat step 2 until the equation contains all of the variables $d_1, \ldots, d_k$.

Step **3**. Match coefficients of atoms left and right to write out linear algebraic equations for $d_1, d_2, \ldots, d_k$. Solve the equations for the unique solution.

Step **4**. The corrected trial solution y with evaluated coefficients $d_1, d_2, \ldots, d_k$ becomes the particular solution y_p.

Symbols

The symbols c_1, c_2 are reserved for use as arbitrary constants in the general solution y_h of the homogeneous equation. Symbols d_1, d_2, d_3, $\ldots$ are reserved for use in the trial solution y of the non-homogeneous equation. Abbreviations: c = constant, d = determined.

Superposition

The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Apply the linear equation **recipe** to find y_h.

(b) Apply **the basic trial solution method** to find y_p.

We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p. Calling d_1, d_2, d_3, $\ldots$ *undetermined* coefficients is misleading, because in fact they are eventually *determined*.

Fixup rule II

The rule predicts the corrected trial solution y without having to substitute y into the differential equation.

- Write down y_h, the general solution of homogeneous equation $ay''+by'+cy=0$, having arbitrary constants c_1, c_2. Create the corrected trial solution y iteratively, as follows.
- Cycle through each term d_pA, where A is a atom. If A is also an atom appearing in y_h, then multiply d_pA and each **related atom** term d_qB by x. Other terms appearing in y are unchanged.
- Repeat until each term d_pA has atom A distinct from all atoms appearing in homogeneous solution y_h. The modified expression y is called the **corrected trial solution**.

Fixup rule III

The rule predicts the corrected trial solution y without substituting it into the differential equation. This iterative algebraic method uses the roots of the characteristic equation to create y.

- Write down the roots of the characteristic equation. Let L denote the list of distinct atoms for these roots.
- Cycle through each term $d_p A$, where A is a atom. If A appears in list L, then multiply $d_p A$ and each **related atom** term $d_q B$ by x. Other terms appearing in y are unchanged.
- Repeat until the atom A in an arbitrary term $d_p A$ of y does not appear in list L.[a] The modified expression y is called the **corrected trial solution**.

[a]The number s of repeats for initial term $d_p A$ equals the multiplicity of the root r which created atom A in list L.

Fixup rule IV

The rule predicts the corrected trial solution y without substituting it into the differential equation. This algebraic method uses the roots of the characteristic equation to create y.

- Write down the roots of the characteristic equation as a list R, according to multiplicity.
- Let G denote a largest group of related atom terms in y with first atom A. If R contains a root r of multiplicity s, and an atom B for r is related to atom A, then multiply all terms of G by x^s. If no root in R has atom related to A, then no action is taken.
- Repeat the previous step for all groups of related atoms in y. The modified expression y is called the **corrected trial solution**.