Name ____________________________ Class Time ______

Math 2250 Extra Credit Problems
Chapter 5
S2015

Submitted work. Please submit one stapled package with this sheet on top. Kindly check-mark the problems submitted and label the paper **Extra Credit**. Label each solved problem with its corresponding problem number, e.g., Xc10.3-20.

Problem XC-bases. (Basis by Computer assist)

Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 3 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 3 & 2 & 2 & 4 & 12 \end{pmatrix}$. Find two different bases for the row space of A, using the following three methods.

1. Pivot columns of A^T.
2. A rowspace computation by computer assist.
3. The **rref**-method: select rows from **rref**(A).

Two of the methods produce exactly the same basis. **Verify** that the two bases $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ are **equivalent**. This means that each vector in $\mathcal{B}_1$ is a linear combination of the vectors in $\mathcal{B}_2$, and conversely, that each vector in $\mathcal{B}_2$ is a linear combination of the vectors in $\mathcal{B}_1$.

Problem XC-Eigenpairs. (Matrix Equations)

Let $A = \begin{pmatrix} -6 & -4 & 11 \\ 3 & 1 & -3 \\ -4 & -4 & 9 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Let P denote a 3×3 matrix. Assume the following result:

> **Lemma 1**. The equality $AP = PT$ holds if and only if the columns $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of P satisfy $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = -2\mathbf{v}_2$, $A\mathbf{v}_3 = 5\mathbf{v}_3$.

(a) Determine three specific columns for P such that $\det(P) \neq 0$ and $AP = PT$. Infinitely many answers are possible.

(b) After reporting the three columns, check the answer by computing $AP - PT$ (it should be zero) and $\det(P)$ (it should be nonzero).

Problem XC5.1-all. (Second order DE)

This problem counts as 700 if section 5.1 was not submitted and 100 otherwise. Solve the following seven parts.

(a) $y'' + 4y' = 0$ (b) $4y'' + 12y' + 9y = 0$ (c) $y'' + 2y' + 5y = 0$
(d) $21y'' + 10y' + y = 0$ (e) $16y'' + 8y' + y = 0$ (f) $y'' + 4y' + (4 + \pi)y = 0$
(g) Find the differential equation $ay'' + by' + cy = 0$, given that e^{-x} and e^{x} are solutions.

Problem XC5.2-18. (Initial value problems)

Given $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$ has three solutions x, $1/x^2$, $\dfrac{\ln|x|}{x^2}$, prove by the Wronskian test that they are independent and then solve for the unique solution satisfying $y(1) = 1$, $y'(1) = 5$, $y''(1) = -11$.

Problem XC5.2-22. (Initial value problem)

Solve the problem $y'' - 4y = 2x$, $y(0) = 2$, $y'(0) = -1/2$, given a particular solution $y_p(x) = -x/2$.

Problem XC5.3-8. **(Complex roots)**
Solve $y'' - 6y' + 25y = 0$.

Problem XC5.3-10. **(Higher order complex roots)**
Solve $y^{iv} + \pi^2 y''' = 0$.

Problem XC5.3-16. **(Fourth order DE)**
Solve the fourth order homogeneous equation whose characteristic equation is $(r-1)(r^3-1) = 0$.

Problem XC5.3-32. **(Theory of equations)**
Solve $y^{iv} - y''' + y'' - 3y' - 6y = 0$. Use the theory of equations [factor theorem, root theorem, rational root theorem, division algorithm] to completely factor the characteristic equation. You may check answers by computer, but please display the complete details of factorization.

Problem XC5.4-20. **(Overdamped, critically damped, underdamped)**
(a) Consider $2x''(t) + 12x'(t) + 50x(t) = 0$. Classify as overdamped, critically damped or underdamped.
(b) Solve $2x''(t)+12x'(t)+50x(t) = 0$, $x(0) = 0$, $x'(0) = -8$. Express the answer in the form $x(t) = C_1 e^{\alpha_1 t}\cos(\beta_1 t - \theta_1)$.
(c) Solve the zero damping problem $2u''(t) + 50u(t) = 0$, $u(0) = 0$, $u'(0) = -8$. Express the answer in phase-amplitude form $u(t) = C_2\cos(\beta_2 t - \theta_2)$.
(d) Using computer assist, display on one graphic plots of $x(t)$ and $u(t)$. The plot should showcase the damping effects. A hand-made replica of a computer or calculator graphic“ is sufficient.

Problem XC5.4-34. **(Inverse problem)**
A body weighing 100 pounds undergoes damped oscillation in a spring-mass system. Assume the differential equation is $mx''+cx'+kx = 0$, with t in seconds and $x(t)$ in feet. Observations give $x(0.4) = 6.1/12$, $x'(0.4) = 0$ and $x(1.2) = 1.4/12$, $x'(1.2) = 0$ as successive maxima of $x(t)$. Then $m = 3.125$ slugs. Find c and k.

Atoms. An **atom** is a power x^n, $n = 0, 1, 2, 3, \ldots$ times a **base atom**. A base atom is one of the terms 1, e^{ax}, $\cos bx$, $\sin bx$, $e^{ax}\cos bx$, $e^{acx}\sin bx$. The symbol n is a non-negative integer. Symbols a and b are real numbers with $b > 0$. Any list of distinct atoms is linearly independent.

Roots and Atoms. Define **atomRoot**(A) as follows. Symbols α, β, r are real numbers, $\beta > 0$ and k is a non-negative integer.

atom A	**atomRoot**(A)
$x^k e^{rx}$	r
$x^k e^{\alpha x}\cos\beta x$	$\alpha + i\beta$
$x^k e^{\alpha x}\sin\beta x$	$\alpha + i\beta$

The fixup rule for undetermined coefficients can be stated as follows:

> *Compute* **atomRoot**(A) *for all atoms A in the trial solution. Assume r is a root of the characteristic equation of multiplicity k. Search the trial solution for atoms B with* **atomRoot**$(B) = r$*, and multiply each such B by x^k. Repeat for all roots of the characteristic equation.*

Problem Xc5.5-1A. **(AtomRoot Part 1)**

1. Evaluate **atomRoot**(A) for $A = 1$, x, x^2, e^{-x}, $\cos 2x$, $\sin 3x$, $x\cos\pi x$, $e^{-x}\sin 3x$, x^3, e^{2x}, $\cos x/2$, $\sin 4x$, $x^2\cos x$, $e^{3x}\sin 2x$.

2. Let $A = xe^{-2x}$ and $B = x^2e^{-2x}$. Verify that **atomRoot**$(A) =$ **atomRoot**(B).

Problem Xc5.5-1B. **(AtomRoot Part 2)**

3. Let $A = xe^{-2x}$ and $B = x^2e^{2x}$. Verify that **atomRoot**$(A) \neq$ **atomRoot**(B).

4. Atoms A and B are said to be **related** if and only if the derivative lists A, A', ... and B, B', ... share a common atom. Prove: atoms A and B are related if and only if $\mathbf{atomRoot}(A) = \mathbf{atomRoot}(B)$.

Problem XC5.5-6. (Undetermined coefficients)

Find a particular solution $y_p(x)$ for the equation $y^{iv} - 4y'' + 4y = xe^{2x} + x^2e^{-2x}$. Check your answer with technology.

Problem XC5.5-12. ()

Find a particular solution $y_p(x)$ for the equation $y^{iv} - 5y'' + 4y = xe^x + x^2e^{2x} + \cos x$. Check your answer by technology.

Problem XC5.5-22. (Shortest trial solution)

Report a shortest trial solution y for the calculation of y_p by the method of undetermined coefficients. To save time, do not do any further undetermined coefficients steps.

$$y^v + 2y''' + 2y'' = 5x^3 + e^{-x} + 4\cos x.$$

Problem XC5.5-54. (Variation of parameters)

Solve by variation of parameters for $y_p(x)$ in the equation $y'' - 16y = xe^{4x}$. Check your answer by technology.

Problem XC5.5-58. (Variation of parameters)

Solve by the method of variation of parameters for $y_p(x)$ in the equation $(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1$. Use the fact that $\{x, 1 + x^2\}$ is a basis of the solution space of the homogeneous equation. Apply (33) in the textbook, after division of the leading coefficient $(x^2 - 1)$. Check your answer by technology.

Problem XC5.6-4. (Harmonic superposition)

Write the general solution $x(t)$ as the superposition of two harmonic oscillations of frequencies 2 and 3, for the initial value problem $x''(t) + 4x(t) = 16\sin 3t$, $x(0) = 0$, $x'(0) = 0$.

Problem XC5.6-8. (Steady-state periodic solution)

The equation $x''(t) + 3x'(t) + 3x(t) = 8\cos 10t + 6\sin 10t$ has a unique steady-state periodic solution of period $2\pi/10$. Find it.

Problem XC5.6-18. (Practical resonance)

Use the equation $\omega = \sqrt{\dfrac{k}{m} - \dfrac{c^2}{2m^2}}$ to decide upon practical resonance for the equation $mx' + cx' + kx = F_0\cos\omega t$ when $F_0 = 10$, $m = 1$, $c = 4$, $k = 5$. Sketch the graph of $C(\omega) = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$ and mark on the graph the location of the resonant frequency (if any). See Figure 5.6.9 in Edwards-Penney.

Problem XC-EPbvp-3.7-4. (LR-circuit)

An LR-circuit with emf $E(t) = 100e^{-12t}$, inductor $L = 2$, resistor $R = 40$ is initialized with $i(0) = 0$. Find the current $i(t)$ for $t \geq 0$ and argue physically and mathematically why the observed current at $t = \infty$ should be zero.

Problem XC-EPbvp-3.7-12. (Steady-state of an RLC-circuit)

Compute the steady-state current in an RLC-circuit with parameters $L = 5$, $R = 50$, $C = 1/200$ and emf $E(t) = 30\cos 100t + 40\sin 100t$. Report the amplitude, phase-lag and period of this solution.

End of extra credit problems chapter 5.